ROOT LOCUS TECHNIQUES IN THE LOG S PLANE FOR DESIGN AND ANALYSIS OF FEEDBACK CONTROL SYSTEMS

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Submitted in partial fulfillment of
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ABSTRACT

Root locus plots in the s plane are practically limited to a span of two decades. In problems where some poles are very close to the origin, while other poles or zeros are at very large values of s, sufficient resolution of the locus may not be available in two decades of the s plane. However, root loci can be plotted in the <u>ln s plane</u> using as many decades as desired (and <u>whichever</u> are desired).

Electro-Scientific Industries Corporation manufactures an analog computer, THE ESIAC, which constructs root locus plots in the ln s plane. This paper presents a mathematical analysis of the ln s plane, a simple method of plotting root loci in the ln s plane, and several practical examples demonstrating the technique. Construction of root loci by this method requires the use of one template.

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TABLE OF SYMBOLS AND ABBREVIATIONS

s - the complex variable

σ - the real part of s

j - $\sqrt{-1}$

ω - the imaginary part of s

ln - logarithm to the base e

w - complex variable $\equiv$ ln s

u - the real part of w

v - the imaginary part of w

$|s|$ - the magnitude of s

θ - the argument of s

∞ - infinity

m - parameter (slope)

b - parameter (intercept)

d - the length of the "perpendicular radial"

α - angle of inclination of the general line, Fig. 5

P_1 - template scale reading at point one

P_2 - template scale reading at point two

K - template scale multiplier

l - the number of poles

n - the number of zeros

r - indexing number

σ_A - the centroid of asymptotes

k - open loop gain (transient response form)

P - pole

Z - zero

Π - factor multiplication symbol (as Σ means summation)

$\overrightarrow{S^i_k}$ - root gain sensitivity

q_i - the root in question

L - the loop gain

∂ - partial derivative

$\equiv$ - definition or identity sign

$\overrightarrow{S^i_{z_j}}$ - root-zero sensitivity

z_j - zero in question

$\overrightarrow{S^i_{p_j}}$ - root-pole sensitivity

p_j - pole in question

$\overrightarrow{U}$ - length and phase reference vector for pole and zero sensitivities

K_f - open loop gain (frequency response form)

ϕ - phase shift of compensator

ω_i - imaginary part of dominant root q

p - pole of compensator

z - zero of compensator

R(s) - LaPlace Transform of input function

C(s) - LaPlace Transform of output function

G(s) - LaPlace Transform of open loop transfer function

C(t) - time response of output

C_i - constants

u(t) - unit step function

I

PLOTTING IN THE LOG S PLANE

1. Introduction

The value of root locus plotting is limited in those applications where some poles and zeros are widely separated and others are close to the origin or close to each other. When scale is expanded to show sufficient detail close to the origin, other poles are moved off the paper. The designer needs to see all the poles and zeros, yet retain sufficient detail near the origin.

Logarithmic scaling solves the problem. This paper examines the properties of the ln s plane and then shows how each root locus sketching/plotting technique in linear coordinates (s plane) also applies in the ln s plane. In the s plane, all the present root locus technique is based upon straight lines, slopes, and lengths. Hence, an analogous procedure requires the ln s plane equivalent of a straight line, slope, and length.

2. The ln s plane

The s plane is defined by the rectangular coordinates $s = \sigma + j\omega$ and the polar coordinates $s = |s| \angle \theta$. The ln s plane, hereinafter denoted by w, shall be defined by rectangular coordinates $w = u + jv$. The conformal mapping relation between the planes is then $w = u + jv = \ln(s) = \ln |s| + j\theta$. Hence, equating reals and imaginaries:

$$\boxed{\begin{aligned} u &= \ln |s| \\ v &= \theta \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} |s| &= e^{u} \\ \theta &= v \end{aligned}}$$

Thus the ordinate of the w plane measures θ and the abscissa measures $\ln |s|$. Referring to Fig. 1, the w plane, note that θ extends from $-\infty$ to $+\infty$, corresponding to more than one polar coordinate revolution in the s plane. Therefore, every point on the s plane is represented, in one to one correspondence in the w plane, in the infinite strip bounded by $0 \leq \theta \leq 2\pi$. Hence, this strip provides single valued correspondence.

The scales of θ and $\ln |s|$ are equal. This choice was made to retain the properties of conformal mapping. The abscissa is linearly calibrated in $\ln |s|$ with one unit of $\ln |s|$ equal in length to one radian of the ordinate. Then a logarithmic scale division calibrates the abscissa directly in $|s|$. This is a most important point; for by using four-cycle semi-log paper, any four decades of θ vs $|s|$ may be plotted. The cross-hatched area on Fig. 1 depicts such an area, and corresponds to the cross-hatched area on the s plane shown in Fig. 2.

Fig. 3 shows this same area on a full-sized sheet of semi-log paper. The horizontal lines (θ = constant) represent radial lines through the origin of the s plane; the vertical lines ($|s|$ = constant) must represent circles around the origin of the s plane. In the s plane, any radial line can be calibrated in length by the family of circles about the origin, $|s| = 1, 2, \ldots \infty$. Hence, the horizontal lines in the w plane are calibrated in length by the vertical lines, $|s| = 1, 2, \ldots \infty$ Summarizing, the semi-log paper provides all the s plane radial lines, already calibrated in length. Further, the slope of these radial lines is the ordinate scale.

To analyze the general line, not passing through the origin, examination of the relation between Cartesian coordinates, σ, ω, and u, v,

shows that:

$$w = \ln(s)$$

$$s = e^{w} = e^{u+jv} = e^{u}e^{jv}$$

$$\sigma + j\omega = e^{u}e^{jv} = e^{u}(\cos v + j\sin v)$$

Equating real and imaginary terms:

$$\boxed{\begin{array}{ll} \sigma = e^{u}\cos v & \omega = e^{u}\sin v \\ \ln(\sigma) = u + \ln\cos v & \ln(\omega) = u + \ln\sin v \end{array}}$$

These relations define families of transcendental curves representing the vertical and horizontal lines of the s plane. Fig. 4 shows these families in the w plane, covering the same four decades as Figs. 1, 2, and 3. The w plane vertical and horizontal lines have been omitted for clarity, but the scales are present. Selected members of the family have been labeled in "s plane counterpart" notation for easy reference.

Notice that the form of all the curves is identical. Examine the three curves: $\sigma = -0.5$, $\sigma = -1$, $\sigma = -2$. Change in the value of σ shifts the curve laterally. Reference to the defining equation shows this effect. Here, ln σ is an arbitrary constant. Changing its value changes the value of u, equally, throughout the length of the curve. Next examine the curves $\omega = +1$ and $\omega = +2$. The same lateral shift occurs as between $\sigma = -1$ and $\sigma = -2$. This observation is expected from the defining equation $\ln \omega = u + \ln \sin v$ and hence, $u = \ln \omega - \ln \sin v$. This effect is generalized below.

At this point it is well to note that a term ln σ with σ taking on

negative values is really not a cause for concern. In the s plane, $\ln(+\sigma)$ is a positive distance measured along the $\theta = 0$ line and $\ln(-\sigma)$ is a positive distance measured along the $\theta = -180^\circ$ line, providing the correct minus sign.

Before moving to the general line, $j\omega = m\sigma + b$, some helpful geometric properties should be examined. In the s plane, the family of lines $\omega = -\infty \ldots -1, 0, +1, 2, \ldots +\infty$ are all symmetric about the line $\sigma = 0$ which is the $j\omega$ axis. The family of lines $\sigma = -\infty \ldots -1, 0, +1, 2, \ldots +\infty$ are all symmetric about the line $\omega = 0$, the horizontal axis. Thus a line of symmetry is a line drawn from the origin, perpendicular to the line in question. Since angles are preserved in conformal mapping, we expect these same families of lines (e.g. $\sigma = 0, -1, \ldots -\infty$) to be symmetrical about their respective perpendicular radials (e.g. $\omega = 0$) in the w plane. Reference to Fig. 4 provides confirmation.[1]

3. The general line

Let the general line (not through the origin) in the s plane be $\omega = m\sigma + b$. Fig. 5 shows such a line. Define the perpendicular from the origin to this line: "the perpendicular radial," and the length of the perpendicular radial "d." The radial parallel to $\omega = m\sigma + b$ (i.e. $\omega = m\sigma$) is defined: "the parallel radial."

[1] Note: The line ($\omega = 0$) is the same line as ($\theta = 0, 180^\circ, 360^\circ$)

$$w = m\bar{v} + b$$

$$e^{u} \sin v = m e^{u} \cos v + b$$

$$e^{u}(\sin v - m \cos v) = b$$

but $m = \tan\alpha$, where α = angle of inclination of line

$$e^{u}\left(\sin v - \frac{\sin\alpha}{\cos\alpha}\cos v\right) = b$$

$$e^{u}(\sin v \cos\alpha - \sin\alpha \cos v) = b \cos\alpha$$

$$e^{u}(\sin(v-\alpha)) = b \cos\alpha$$

$$u + \ln \sin(v-\alpha) = \ln(b \cos\alpha)$$

But $u + \ln \sin v = \ln w$ is the transformation equation for the Cartesian coordinate lines. Hence: the transcendental curves corresponding to all the straight lines in the s plane, not passing through the origin, have the same form as the Cartesian coordinate (vertical and horizontal) lines. Hence, the shape is the same, and they differ only in positioning. Therefore, one template may be used to draw any non-radial straight line provided adequate rules for positioning it are developed.

Fig. 6 illustrates such a template. The template represents every non-radial straight line, mapped onto w, just as surely as the edge of a triangle or T-square represents every straight line in the s plane.

For clarity, rules for positioning the template, reading slope, and reading length are listed now, with proofs to follow in the next section:

a. The template is always used in the same orientation, with the

line of symmetry horizontal. (Place template on a T-square and position vertically and horizontally, keeping the orientation the same.)

b. The slope of any line corresponds to the template slope lines, read on the ordinate scale of the w plane. There are two such slope lines, always 180 degrees apart. These correspond to "sense" on the line.

c. To draw any line, knowing one point and slope, slide template vertically until slope lines match required slope on ordinate; then, slide horizontally until known point is on the template.

d. To connect any two points with a "straight line:" plot these points on the w plane ($\theta,\ |s|$). Then slide vertically and horizontally until both points are on the template, and draw the line. (Line of symmetry and slope lines must remain horizontal--you are positioning, not orienting the template.) The template will only coincide with the two points at one place, since there is only one straight line between two points. The slope of this line can be read at the slope lines.

e. Where "sense" is important, the 180 degree slope ambiguity is solved as follows: When connecting point one to point two, follow the template line from point one to point two and continue in that direction to the correct slope line. This line then reads the slope from point one to point two.

f. The template is calibrated for length. The distance between any two points on the line is equal to $(p_1 - p_2)K$, where p_1 and p_2 are the template scale readings at point one and point two respectively:

and K is a scale multiplier, varying with the length of the perpendicular radial. Values of K can be read directly on the $|s|$ scale at the template arrow.

Using this template, straight lines, slopes, and lengths can be plotted as quickly on the w plane as a ruler and protracter plot them on the s plane.

4. Justification (casual reader may skip)

The w plane equation of the general straight line is

$$u + \ln(\sin(v-\alpha)) = \ln(b\cos\alpha)$$

$$u = \ln(b\cos\alpha) - \ln(\sin(v-\alpha))$$

Recall: $u = \ln\omega - \ln(\sin v)$ was an equation of cartesian coordinates, and changes in ω shifted the curve laterally only. In the general equation, $\ln(b\cos\alpha)$ corresponds to $\ln\omega$. (b and α are constants for a given line.) Changing b will have the same qualitative effect as changing ω, i.e. shifting the curve laterally.

Referring to Fig. 5, note $b\cos\alpha = d$. Hence:

$u = d - \ln(\sin(v-\alpha))$. It has just been shown how, by moving b, d is changed and the curve is shifted horizontally. Now, hold d fixed and change the slope (or α). Recall the Cartesian defining equations for vertical and horizontal lines:

$$u = \ln\sigma - \ln(\cos v) \quad \text{and}$$

$$u = \ln\omega - \ln(\sin v) = \ln\omega - \ln\left(\cos\left(v - \frac{\pi}{2}\right)\right)$$

When d is held fixed and these two are compared, $\sigma = \omega$, and the only difference is the α term $\alpha = 0$ and $\alpha = +\pi/2$

And it has been previously shown in Fig. 4 that this corresponded to a vertical shift of 90 degrees only. Hence, varying any parameter shifts

the curves position only.

Reference to Fig. 5 shows that as $|s|$ increases to large values, the slope of s approaches the slope of the general line. Since we plot $|s|$ vs θ in w, we expect the general line to become asymptotic to the parallel radial at large values. The parallel radials are 90 degrees (looking both directions) on either side from the perpendicular radial. And since the slope of the general line is equal to the slope of the parallel radial, we need only to read the slope of the parallel radial. Since all radials are the horizontal lines of the w plane semi-log paper, the ordinate scale is used.

Consider the concept of length in the s plane. Given the correct length calibration of all lines passing through the origin, it is possible to calibrate any other line. Draw the parallel radial $\omega = m\sigma$ corresponding to the general line $\omega = m\sigma + b$. Then mark off equal lengths on the parallel radial ($-\infty \ldots -2, -1, 0, +1, +2, \ldots \infty$) and erect perpendiculars to the parallel radial at these equally spaced points. This family of perpendiculars will then calibrate $\omega = m\sigma + b$ in length. Now note that the semi-log graph paper provides all the radials pre-calibrated for distance in the w plane. Then, just as the family of lines $\omega = -\frac{1}{m}\sigma + b$, with b the generating parameter, will calibrate $\omega = m\sigma + b$ in the s plane; so will this family, mapped into the w plane, calibrate $\omega = m\sigma + b$, mapped into the w plane.

Since the spacing between the calibrating family is not constant, but changes with ln $|s|$, length is not as simple in the w plane as in the s plane.

A single calibrated template is desired, not a calibrating family of

curves. Hence, the template may be arbitrarily calibrated in length corresponding to length along the line $\nabla = -1$, using the family $\omega = -\infty \ldots -2, -1, 0, +1, +2, \ldots + \infty$; and a correction factor (multiplier) is used for other lines.

In both the s plane and the w plane, the general line is orthogonal to and symmetric about the perpendicular radial, and the calibrating family of lines is orthogonal to and symmetric about the parallel radial. This fact is invariant with changes in slope. Hence, for a given d (the length of the perpendicular radial) the general line and its calibrating family and their corresponding lines of symmetry are always 90 degrees apart. On the w plane, Fig. 4, it can be seen that change in slope, with fixed d, merely moves the general line and its calibrating family up and down. <u>There is no relative movement between the two</u>. Hence, for a given d, one calibration is valid for any slope. d = 1 has been chosen for calibration of the template in Fig. 6.

Thus, to make the calibration valid for any non-radial line, only correction for changes in d is needed. As parameter d changes [$u = d - \ln(\sin(v - \alpha)$], the general line moves laterally, while the calibrating family remains fixed--changing the calibration.

The scaling function results from two important properties of the w plane:

a. All radials are scaled logarithmically to read in distance. Hence, the horizontal distance from one to two is equal to the distance from five to ten; or a horizontal distance d is a constant multiplier, regardless of position on the radial.

b. The horizontal distance between any two members of the family of calibrating curves is constant regardless of slope; i.e. $\omega = 1$

and $\omega = 2$ have the same horizontal separation in the w plane at any ∇ .

Refer to the defining equations:

$$u = \ln \omega - \ln \sin v$$

$$u_1 - u_2 = (\ln \omega_1 - \ln \sin v) - (\ln \omega_2 - \ln \sin v)$$

$$u_1 - u_2 = \ln \omega_1 - \ln \omega_2 \qquad \text{(not a function of v)}$$

Hence: as the general line moves horizontally with changing d, each point on the line crosses an equal number of calibrating curves. Hence, length calibration changes by an equal amount anywhere along the line.

In the limit, the general line coincides with the parallel radial. Hence, horizontal movement (increase in d) by one octave will change the value of the calibration of the general line's extremities by one octave; hence, changes the entire line calibration by one octave. This movement corresponds to a change of d from one to two. Therefore, d can be used as a scale calibration multiplier for the whole line. d is defined as the distance from origin to vertex in the w plane, hence can be read at the arrow on the template.

5. Plotting root loci

Root loci may now be drawn in the w plane. Each sketching aid in the s plane is based on straight lines and their slopes and lengths. All these are now available in the w plane.

Logarithmic root loci must be plotted on semi-log paper with a specific relation between ordinate and abscissa scale divisions, to preserve conformal mapping: namely, one cycle equals 131.8 degrees. Such semi-log paper is commercially available through Electro-Scientific Industries Corpn., 7524 Southwest Macadam Avenue, Portland 19, Oregon;

designated as ESIAC Form 44. This paper is four-cycle, two and one-half inches per cycle by 6.83 inches per 360 degrees, with five degree divisions and 30-degree heavy divisions.

Plot root loci as follows:

a. Choose and label the four abscissa ($|s|$) decades of interest, e.g., .01, 0.1, 1, 10, 100.

b. Label ordinates, zero to 360 degrees.

c. Convert poles and zeros to magnitude and angle form, and plot. (Poles and zeros <u>at</u> origin can not, and need not be plotted.)

d. Sketch root locus segments on real axis (θ = 0, 180°, 360°). (Start with pole or zero most positive and trace the real axis from that location toward minus infinity. Those segments of the real axis which are on the negative side of an odd number of poles (or zeros plus poles) are root loci.) Care must be taken in this matter since this means tracing left on the positive real axis and right on the negative real axis. Furthermore, poles at the origin must be counted when crossing from positive to negative real axis. When k is negative, substitute "even" for "odd" in this rule.

e. Count poles, l , and count zeros, n. Then l-n root locus segments extend to infinity.

f. Locate and plot centroid of asymptotes on the real axis,

$$\sigma_A = \frac{(\Sigma \text{Poles} - \Sigma \text{zeros})}{l - n}$$

g. Angles of asymptotes are found as follows:

$$\left.\begin{aligned} &(1)\ \text{For positive k:}\quad \text{Angles} = \frac{(2r+1)\pi}{|l-n|} \\ &(2)\ \text{For negative k:}\quad \text{Angles} = \frac{2\pi r}{|l-n|} \end{aligned}\right\} r = 1, 2, \ldots (|l-n|-1) \text{ radians}$$

To plot asymptotes, place template slope line coincident with graph paper horizontal line, $\Theta =$ required slope, slide horizontally until template coincides with ∇_A and draw asymptote. Repeat until all asymptotes are plotted.

h. Angle of emergence from a complex pole may be calculated just as in linear theory. Keeping template slope lines horizontal, slide template until it connects the complex pole/zero with another pole/zero. Read slope of connecting line at template slope line and record. Repeat for all poles and zeros. Then angle of emergence equals 180 degrees - Σ (recorded slopes) for positive k, and 360 degrees - Σ(recorded slopes) for negative k. Place template slope line on this calculated value, slide over to complex pole/zero, and draw a short segment from pole or zero in template direction. This segment is the emergence from the pole.

i. Sketch in locus. At any questionable points where accuracy is required, use the template as a Spirule; i.e., assume a trial point; measure angles from all poles and zeros (as is shown above) with the template. Then, if the sum of the angles equals 180 degrees (for positive k, or 360 degrees for negative k), the point is on the locus. If not, another trial is needed. Direction of new trial point is indicated by whether the sum of the angles is greater than or less than 180 degrees.

j. Root and gain point location on the locus is analogous to linear methods; i.e., $k = \dfrac{\prod |s+p|}{\prod |s+z|}$ of the loop transfer function. Use template to measure distance from each pole/zero to point on locus in question and apply above equation. (Connect two points with template; length = $(p_2 - p_1)K$ where $K = |s|$ as read at

II

APPLICATION TO SENSITIVITY DESIGN

1. Sensitivity design in the s plane

Gain and root sensitivity are important considerations in control system design and analysis. An exact method with no approximations has been recently developed by Rung and Thaler (2). This method uses root locus plot in the s plane which is adaptable to the ln s plane. The balance of this section is devoted to explanation of the method, definitions, and adaptation to the w plane. Reference (2) contains the proofs of the relationships between definitions to follow.

Sensitivities are defined as vector quantities.

Eq. 2.1 -- Root-Gain Sensitivity $\overrightarrow{S_k^i} \equiv \overrightarrow{\frac{\partial q_i}{\partial k/k}} = \overrightarrow{\frac{-1}{\left.\frac{\partial L}{\partial s}\right|_{s=q_i}}}$

where q_i is the root in question and L is loop gain. Thus Gain Sensitivity $\equiv$ change in root position per percent change in gain. (Since parameter changes may move the root in question in different directions, sensitivities must be vector quantities.)

Eq. 2.2 -- Root-Zero Sensitivity $\overrightarrow{S_{z_j}^i} \equiv \overrightarrow{\frac{\partial q_i}{\partial z_j}} = \frac{\overrightarrow{S_k^i}}{z_j - q_i}$

Eq. 2.3 -- Root-Pole Sensitivity $\overrightarrow{S_{p_j}^i} \equiv \overrightarrow{\frac{\partial q_i}{\partial p_j}} = \frac{\overrightarrow{S_k^i}}{q_i - p_j}$

Pole and zero sensitivities $\equiv$ change in root position per change in pole or zero position. Rung and Thaler (2) also show that the sum of root and pole sensitivities equals one.

Eq. 2.4 -- $\sum_j S_{z_j}^i + \sum_j \overrightarrow{S_{p_j}^i} = 1$ Using this fact along with equations 2.2 and 2.3, vector representations of sensitivities are shown in Fig. 7.

Fig. 7 is a root locus plot and sensitivity vector diagram of the open loop transfer function $G(s) = \frac{k(s^2+3s+3)}{s(s+1.7)(s+3)}$ in the s plane; k was chosen equal to 2.42, corresponding to a root at q_1 as shown.

The following construction procedure was used. Draw a vector from q_1 toward each pole with length equal to (1/distance from q_1 to pole). These vectors define pole sensitivities, $\overrightarrow{S^1_{p_0}}$, $\overrightarrow{S^1_{p_1}}$, $\overrightarrow{S^1_{p_2}}$. Draw a vector from q_1 away from each zero, defining zero sensitivities, $\overrightarrow{S^1_{z_1}}$, $\overrightarrow{S^1_{z_2}}$. Form the vector sum of $\overrightarrow{S^1_{p_0}}$, $\overrightarrow{S^1_{p_1}}$, $\overrightarrow{S^1_{p_2}}$, $\overrightarrow{S^1_{z_1}}$, $\overrightarrow{S^1_{z_2}}$ and call if $\vec{U}$. <u>Then $\vec{U}$ is the reference for length and phase</u> of the pole and zero sensitivity vectors $\overrightarrow{S^1_{p_0}}$, $\overrightarrow{S^1_{p_1}}$, $\overrightarrow{S^1_{p_2}}$, $\overrightarrow{S^1_{z_1}}$, $\overrightarrow{S^1_{z_2}}$. Phase is measured <u>clockwise</u> from $\vec{U}$. For example, $\overrightarrow{S^1_{p_1}} = \left|\frac{\text{length of } S^1_{p_1}}{\text{length of } \vec{U}}\right| \angle \alpha$ or $\overrightarrow{S_{p_1}} = .585\angle 76^\circ$

Gain sensitivity $\overrightarrow{S^i_k}$ is directionally coincident with $\vec{U}$ which is always tangent to the root locus. (When gain changes, root must move along root locus.) Magnitude of the $\overrightarrow{S^i_k}$ vector is found from the relationship:

$$\overrightarrow{S^i_{z_j}} = \frac{\overrightarrow{S^i_k}}{q_i - p_j} \qquad \text{or} \qquad \overrightarrow{S^i_k} = \overrightarrow{S^i_{p_j}}\,(\overrightarrow{q_i - p_j})$$

When the pole is chosen at the origin, $\overrightarrow{S^i_k} = \overrightarrow{S^i_{p_0}}\,(+q_i)$

In this example $-q_1 = \overrightarrow{1.225}\angle 157^\circ$

$$q_1 = 1.225\angle -23^\circ\ , \qquad \overrightarrow{S^1_{p_0}} = \overrightarrow{\frac{.82}{2.05}\angle 320^\circ} = \overrightarrow{.4\angle 320^\circ}$$

$$\overrightarrow{S^i_k} = (\overrightarrow{.4\angle 320^\circ})(\overrightarrow{1.225\angle -23^\circ}) = \overrightarrow{.49\angle 297^\circ}$$

(measured counterclockwise as $\vec{U}$ is measured).

Since sensitivities are measured relative to $\vec{U}$, the designer can manipulate the poles and zeros of compensation devices to achieve optimum placement of $\vec{U}$, in addition to stability and response considerations. This technique is demonstrated clearly by Rung and Thaler (2).

2. Sensitivity design in the ln s plane

For use in the w plane, this method is most practical and rapid using a separate graph for vector diagrams. Since straight line segments of given length and slope can be drawn with the template, vector diagrams can be drawn on the w plane, but the method is slower and gains no advantage. After root locus and root position (or desired root position) q_i have been plotted, use the template to measure and record distance and phase angle from q_i to each open loop pole and zero. Then the vector diagram can be constructed on separate linear coordinates. The following example problem is taken from Rung and Thaler (2), exactly, but performed in the w plane to demonstrate the entire method.

3. Design example

The plant to be compensated has the open loop transfer function

$$G(s) = \frac{K_f}{s(.7s+1)(.3s+1)} = \frac{k}{s(s+1.43)(s+3.33)} \quad \text{where}$$
$$K_f = .21k$$

Dynamic bandwidth requirements lead to the desired location of the dominant roots at $-q = -0.2 \pm j0.35 = 0.4\angle 120^\circ$, that is, $q = 0.4\angle -60^\circ$. All three plant poles are subject to fluctuations. It is desired to design a cascade compensator, satisfying the above dominant root requirement, and in addition, guaranteeing a minimum value of sensitivity of q to the poles' fluctuations. Fig. 8 shows plant singularities, P_0, P_1, P_2, and desired root location, q_1. Using the template, measure and record distance

and phase angle from each singularity to the desired root position. This information will first, determine the phase angle needed in the compensator to force the root locus through the desired root position, and secondly allow the vector diagram to be drawn. From Fig. 8:

Singularity	Phase	Distance	1/Distance
p_0	120	0.4	2.5
p_1	16	1.27	0.787
p_2	6	3.09	0.323
	142		

Phase angle needed from compensator is thus $-180^\circ + 142^\circ = -38^\circ$, indicating a phase lag filter. Now we may draw the vector diagram (Fig. 9) assuming q_i is on the root locus (remembering vectors are drawn from roots toward poles). Note that the compensator <u>will</u> <u>not</u> <u>change</u> the pole sensitivity vectors, but will rather add to their vector sum and change $\vec{U}$, the reference. On Fig. 9 the vector sum of the sensitivity vectors is drawn, $\vec{QI}$. $\vec{QI}$ is the uncompensated $\vec{U'}$ vector. The sensitivity vectors resulting from the pole and zero of the compensator will vectorially add to $\vec{U'}$, and generate the locus of $\vec{U}$ as they are moved along the $-j\omega$ axis (always with correct separation to provide -38° phase lag). Rung and Thaler (2) show this locus to be a circle with center at I, the tip of the "uncompensated $\vec{U}$ vector," with radius $r = \frac{1}{\omega_1} \sin \phi$, where $\phi = 38^\circ$ (the phase shift of the compensator), and $\omega_1 = 0.35$ (the imaginary part of the dominant root q_i).

Stepwise: the compensator pole will have a sensitivity vector of length 1/QP where QP is the distance from pole P to Q. As P moves along the negative real axis, 1/QP varies from a maximum when P is perpendicularly below Q in the s plane to a minimum when P is at infinity. Added

vectorially to $\overrightarrow{U}'$, $\overrightarrow{1/QP}$ generates a circle of radius $R = 1/2\omega_1 = 1/0.7 = 1.43$, with center vertically below I. This circle is drawn on Fig. 9 and labeled the "M" circle. Then the sensitivity vector generated by the compensator zero, $\overrightarrow{1/QZ}$, is added vectorially to the "M" circle, resulting in the U circle, with $r = \frac{1}{\omega_1} \sin \phi$, center at I (proof in reference two). The U circle is drawn and labeled in Fig. 9.

In this example, minimum pole sensitivity has been specified, hence the longest possible $\overrightarrow{U}$ vector is desired. $\overrightarrow{QU}$, the intersection of the U circle with the extension of $\overrightarrow{QI}$, is this vector (desired $\overrightarrow{U}$). Now, to find P and Z from $\overrightarrow{U}$, proceed as follows: Draw $\overrightarrow{OJ}$ perpendicular to $\overrightarrow{QU}$. Measure arcs $JN = JM = \phi$. Then, the direction of $\overrightarrow{IN}$ is the direction of QZ (i.e., 208°), and the direction of $\overrightarrow{IM}$ is the direction of $\overrightarrow{QP}$ (i.e., 246°). (Proof in reference two.)

Using these directions and the template, P and Z are located on the -180° axis in Fig. 8. ($P = 0.9\angle{-180^\circ}$ and $Z = 0.36\angle{-180^\circ}$.)

4. Summary

This method has designed the compensator to force the dominant roots to the desired point and ensured minimum root sensitivity for cascade compensation. Furthermore, the sensitivity of each pole is known. Sensitivities are tabulated below.

$\overrightarrow{S'_{p_0}} = 0.61\angle 335^\circ$ $\qquad \overrightarrow{S'_p} = 0.19\angle 27.5^\circ$ (compensator)

$\overrightarrow{S'_{p_1}} = 0.192\angle 77.5^\circ$ $\qquad \overrightarrow{S'_z} = 0.093\angle 65^\circ$ (Compensator)

$\overrightarrow{S'_{p_2}} = 0.088\angle 87.5^\circ$ $\qquad S'_K = 0.249\angle 273^\circ = S_{p_0}$ (/q.1)

$\qquad \overrightarrow{U} = 4.1\ \angle 273^\circ$

Sensitivity design is obviously simpler in the s plane than in the w plane. But when a root locus can not be handled in the s plane, due to widely spread open loop poles and zeros, or when desired root location is very near the origin, this w plane technique does work. There is no loss in usefulness or generality caused by plotting the <u>vector</u> diagram on linear coordinates. Choose the scale so that the largest sensitivity vectors plot easily. Then any vectors which are too small to plot accurately can be ignored, since their effect on the $\vec{U}$ vector will be negligible, and small sensitivities are desirable.

III

COMPENSATION DESIGN EXAMPLE

1. Compensation

This example demonstrates:

a. Clarity of w plane root locus analysis for compensator design where a pole/zero pair is very close to the origin and other poles and zeros are far removed.

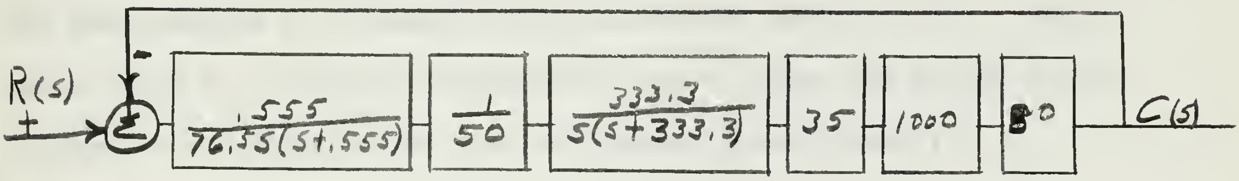
b. Relationship of frequency and transient response to w plane root locus.

c. Breakaway-point calculation by plot of k versus real s (Wheeler Plot) directly on root locus plot.

d. Graphical calculation of residues.

e. Direct reading scale of damping factor of dominant roots on w plane root locus plot.

A stabilization-type servo is designed as an example by Thaler and Brown (1). Compensation of the roll-stabilization loop in the w plane follows:



The uncompensated open loop transfer function is:

$$G(s) = \frac{135000}{s(s+.555)(s+333)}$$

A symmetrical compensator is chosen to meet the requirement of preserving the gain. Trial and error methods lead to a trial compensator of

$$\frac{(s+0.55)(s+2.17)}{(s+0.055)(s+21.7)}$$

The compensated system then has an open loop transfer function:

$$G(s) = \frac{135000(s+.55)(s+2.17)}{s(s+.055)(s+.555)(s+21.7)(s+333)}$$

Fig. 10 is a root locus plot of the uncompensated system in the w plane, showing the closed loop complex roots corresponding to k = 135,000, and the instability of the system; the phase margin is zero. The locus breakaway point was determined by plotting gains versus real s. This plot can be made directly on the root locus plot, as shown.

Fig. 11 is the w plane plot of the root locus of the compensated system, plotted over eight decades. The superiority of the w plane is clearly demonstrated here, in its property of showing all the poles and zeros, regardless of separation, and the complete locus.

The complex closed loop root corresponding to k = 135,000 is shown. The phase margin is 30 degrees, and the damping factor is 0.5. Since phase angle is a vertical coordinate in the w plane, the cosine may be plotted as a vertical scale, to read damping factor directly.

2. Frequency response

Fig. 12 and Fig. 13 show the Bode diagrams corresponding to Fig. 10 and Fig. 11, respectively. Corner frequencies occur in the same position on the Bode diagram as the open loop poles and zeros appear in the root locus. The frequency response can be calculated directly from the root locus in the w plane, in a manner analogous to that in the s plane. However, the standard approximate methods are much faster and should be used,

unless accurate detail is required.

3. Transient response

The transient response may be calculated directly from the w plane root locus plot by calculation of the residues.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{135000(s+.55)(s+2.17)}{135000(s+.55)(s+2.17)+s(s+.055)(s+.555)(s+21.7)(s+333)}$$

Using roots found by Fig. 11 to factor the denominator,

$$\frac{C(s)}{R(s)} = \frac{135000(s+.55)(s+2.17)}{(s+.552)(s+2.42)(s+334.2)(s+9.6+j16.6)(s+9.6-j16.6)}$$

For a unit step input, $R(s) = \frac{1}{s}$ and

$$C(s) = \frac{135000(s+.55)(s+2.17)}{s(s+.552)(s+2.42)(s+334.2)(s+9.6+j16.6)(s+9.6-j16.6)}$$

The inverse transform is:

$$C(t) = C_0 + C_1 \epsilon^{-.552t} + C_2 \epsilon^{-2.42t} + C_3 \epsilon^{-334.2t} + C_4 \epsilon^{-(9.6+j16.6)t} + C_5 \epsilon^{-(9.6-j16.6)t}$$

Evaluation of $C_0, C_1 \ldots$ is done on the root locus plot by evaluation of the residues. Vectors are drawn from all other roots and zeros to root s = -2.42. Lengths and angles are recorded, and C_2 is evaluated from the equation below. (The root at -0.552 is effectively cancelled by the zero at -0.550 and could have been neglected.) This procedure is repeated for each root and all coefficients are thus evaluated. See below:

$$C_0 = \frac{135000(.55)(2.17)}{(.552)(2.42)(334.2)(561.12)} = +.984 \angle 0^\circ$$

$$C_1 = \frac{135000(-.002)(1.618)}{(-.552)(1.868)(333.64)(55145)} = +0.00356 \angle 0^\circ$$

$$C_2 = \frac{135000(-1.87)(-0.25)}{(-2.42)(-1.868)(331.78)(327.11)} +0.1285 \angle 0^\circ$$

$$C_3 = \frac{135000(-333.65)(-332.03)}{(-334.2)(-333.64)(-331.78)(105640.72)} = -.00382 \angle 0^\circ$$

$$C_4 = \boxed{.662 \angle +147.7^\circ}$$

$$C_5 = \boxed{.662 \angle -147.7^\circ}$$

(When a root is on the real axis, the angular summation will equal π, 2π, or a multiple thereof. In the above calculations, angles of zero or 2π are omitted, and angles of π are indicated by a negative sign in front of the factor. Angular contributions from conjugate roots cancel, and conjugate factors were multipled together before entering above.)

$$C(t) = .984 + .00356\,\epsilon^{-.552t} + .1285\,\epsilon^{-2.42t} - .00382\,\epsilon^{-334.2t}$$
$$+ .662\,\epsilon^{j2.57}\,\epsilon^{(-9.61+j16.6)t} + .662\,\epsilon^{-j2.57}\,\epsilon^{(-9.6-j16.6)t}$$

$$C(t) = .984 + .00356\,\epsilon^{-.552t} + .1285\,\epsilon^{-2.42t}$$
$$- .00382\,\epsilon^{-334.2t} + 1.324\,\epsilon^{-9.6t}\cos(16.6t + 147.7)\,u(t)$$

IV

FACTORING A POLYNOMIAL

1. One real root, one pair of complex-conjugate roots

Any polynomial, regardless of order, may be factored by successive applications of the root locus plot.

The following example demonstrates the factoring of a third-order case:

$$s^3 + 52s^2 + 104s + 200$$

Set equal to zero and manipulate form:

$$s^3 + 52s^2 + 104s + 200 = 0$$

$$s^3 + 52s^2 + 104s = -200$$

$$1 = \frac{-200}{s^3 + 52s^2 + 104s} = \frac{-200}{s(s^2 + 52s + 104)}$$

$$-1 = \frac{200}{s(s^2 + 52s + 104)} = \frac{200}{s(s + 49.9)(s + 2.1)}$$

Closed loop roots of this function are factors of the original polynomial; (see Fig. 14). Roots were found at

$$s = 50\angle 180^\circ = -50 + j0$$

$$s = 2\angle 120^\circ = -1 - j\sqrt{3}$$

$$s = 2\angle 240^\circ = -1 + j\sqrt{3}$$

by methods of Chapter I. Thus the factored form is:

$$s^3 + 52s^2 + 104s + 200 = (s+50)(s+1+j\sqrt{3})(s+1-j\sqrt{3})$$

2. Two pair of complex-conjugate roots

Factor $s^4 + 18s^3 + 87s^2 - 210s + 250$

Set equal to zero and manipulate,

$$s^4 + 18s^3 + 87s^2 - 210s + 250 = 0$$

$$s^4 + 18s^3 + 87s^2 = 210s - 250$$

$$-1 = \frac{-210(s - 1.19)}{s^2(s+9+j2.45)(s+9-j2.45)}$$

Closed loop roots of this function are the desired factors.

To demonstrate the principle of successive root locus plots, an alternate manipulation will be used here in the solution of this fourth-order case. This technique can be applied to any order.

$$s^4 + 18s^3 + 87s^2 - 210s = -250$$

Eq. A. $$-1 = \frac{250}{s^4 + 18s^3 + 87s^2 - 210s} = \frac{250}{s(s^3 + 18s^2 + 87s - 210)}$$

Now, to factor the cubic in the denominator, set it equal to zero and manipulate as before: $s^3 + 18s^2 + 87s = 210$

$$-1 = \frac{-210}{s(s^2 + 18s + 87)} = \frac{-210}{s(s+9+j2.45)(s+9-j2.45)}$$

Closed loop roots of this function are determined by the root locus plot in Fig. 15. These roots are:

$$s = 1.73\angle 0^\circ = 1.73 + j0$$

$$s = 11\angle 153.5^\circ = -9.88 + j4.92$$

$$s = 11\angle 206.5^\circ = -9.88 - j4.92$$

Thus the cubic is factored: $(s-1.73)(s+9.88+j4.92)(s+9.88-j4.92)$

Then a root locus plot is made of Eq. A.

$$-1 = \frac{250}{s(s-1.73)(s+9.88+j4.92)(s+9.88-j4.92)}$$

Fig. 16 shows the closed loop roots to be:

$$s = 1.41 \angle 45^\circ = 1 + j1$$

$$s = 1.41 \angle 315^\circ = 1 - j1$$

$$s = 11.2 \angle 206.5^\circ = -10 - j5$$

$$s = 11.2 \angle 153.5^\circ = -10 + j5$$

Hence, $s^4 + 18s^3 + 87s^2 - 210s + 250 = (s+10+j5)(s+10-j5)(s-1+j1)(s-1-j1)$

For large-order polynomials, manipulations should be made to place the largest factorable (at least quadradic) group of terms in the numerator. Then, if the denominator is still unfactorable, the process is repeated. This procedure will reduce the polynomial size by three orders per plot.

V

CONCLUSION AND APPENDICES

1. Conclusion

Root locus plots may be sketched or constructed in the ln s plane whenever the s plane plot fails to provide satisfactory resolution of the locus. Plotting in the ln s plane is somewhat more tedious and slower than plotting in the s plane, but does provide needed detail, while showing all the poles and zeros and all of the locus of interest. The s plane should be used in preference to the ln s plane whenever it is adequate.

This paper lists a straightforward, step-by-step procedure for plotting in the ln s plane, and includes mathematical proof of the theory. The method is demonstrated in three practical examples in Chapters II, III, and IV.

Root locus plots in the ln s plane provide another useful tool in the design and analysis of feedback control systems.

Figure 1

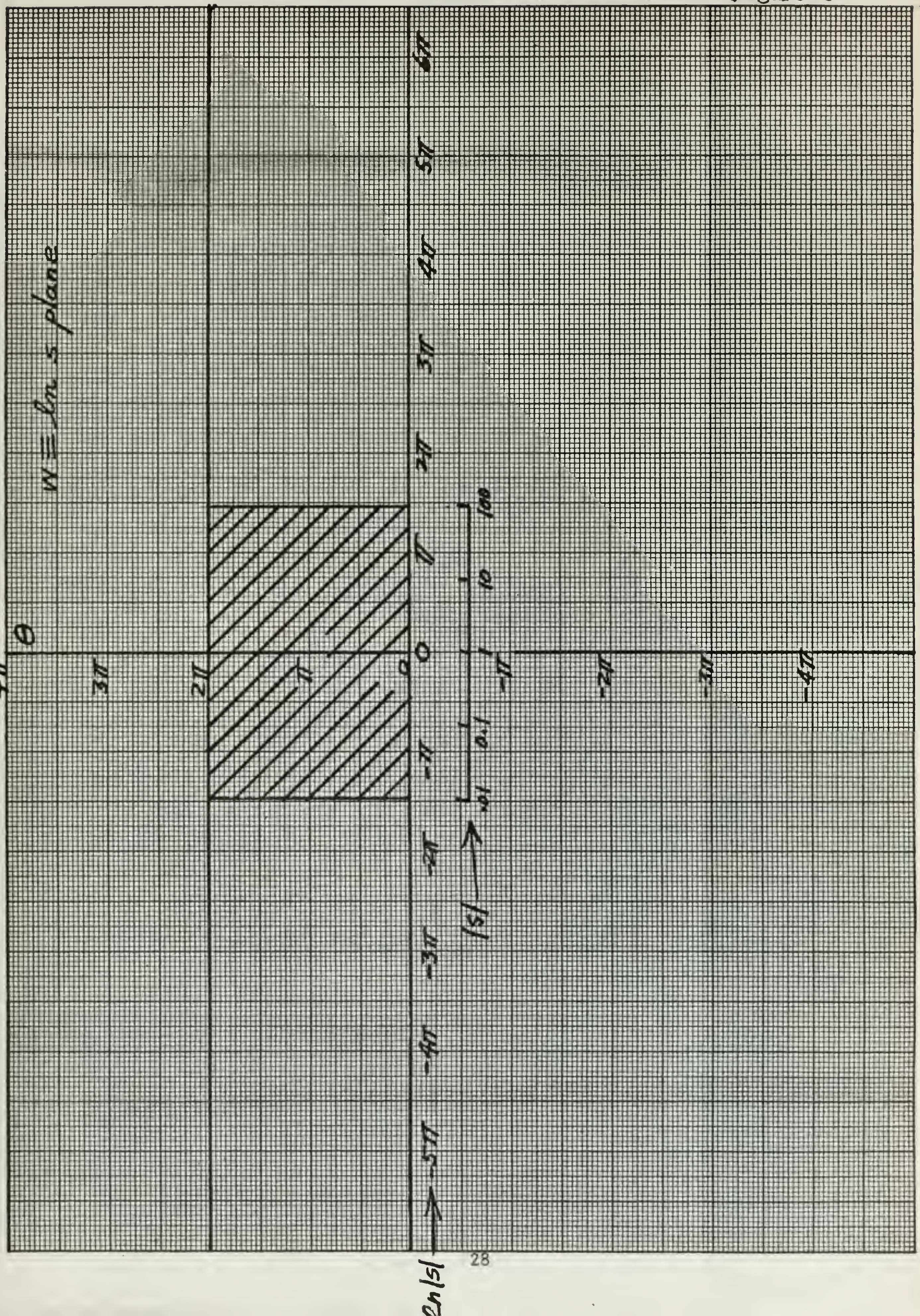
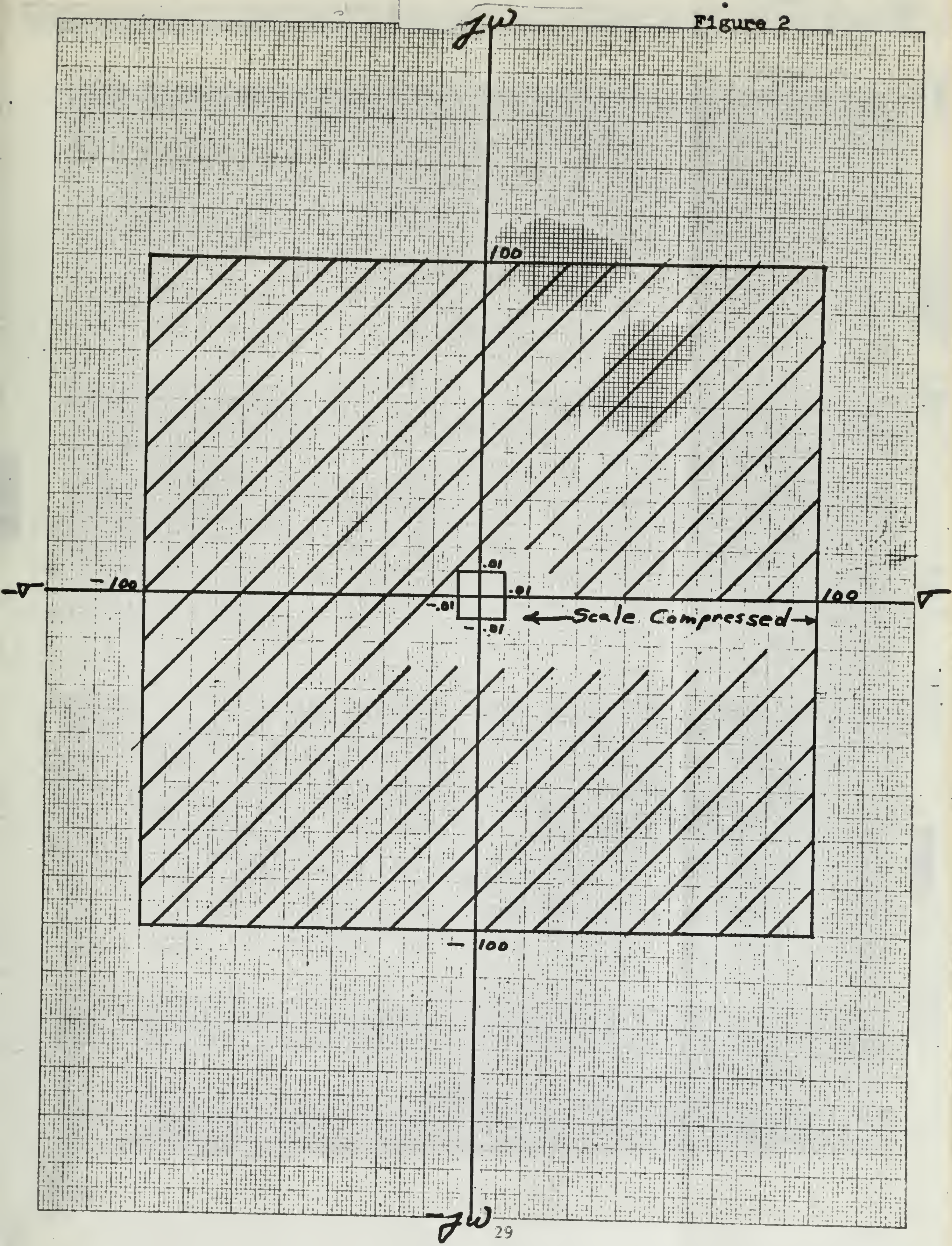

W ≡ ln s plane
θ
4π
3π
2π
π
0
-π
-2π
-3π
-4π
ln|s|
-5π
-4π
-3π
-2π
-π
0
π
2π
3π
4π
5π
6π
|s|
.01
0.1
1
10
100

Figure 2



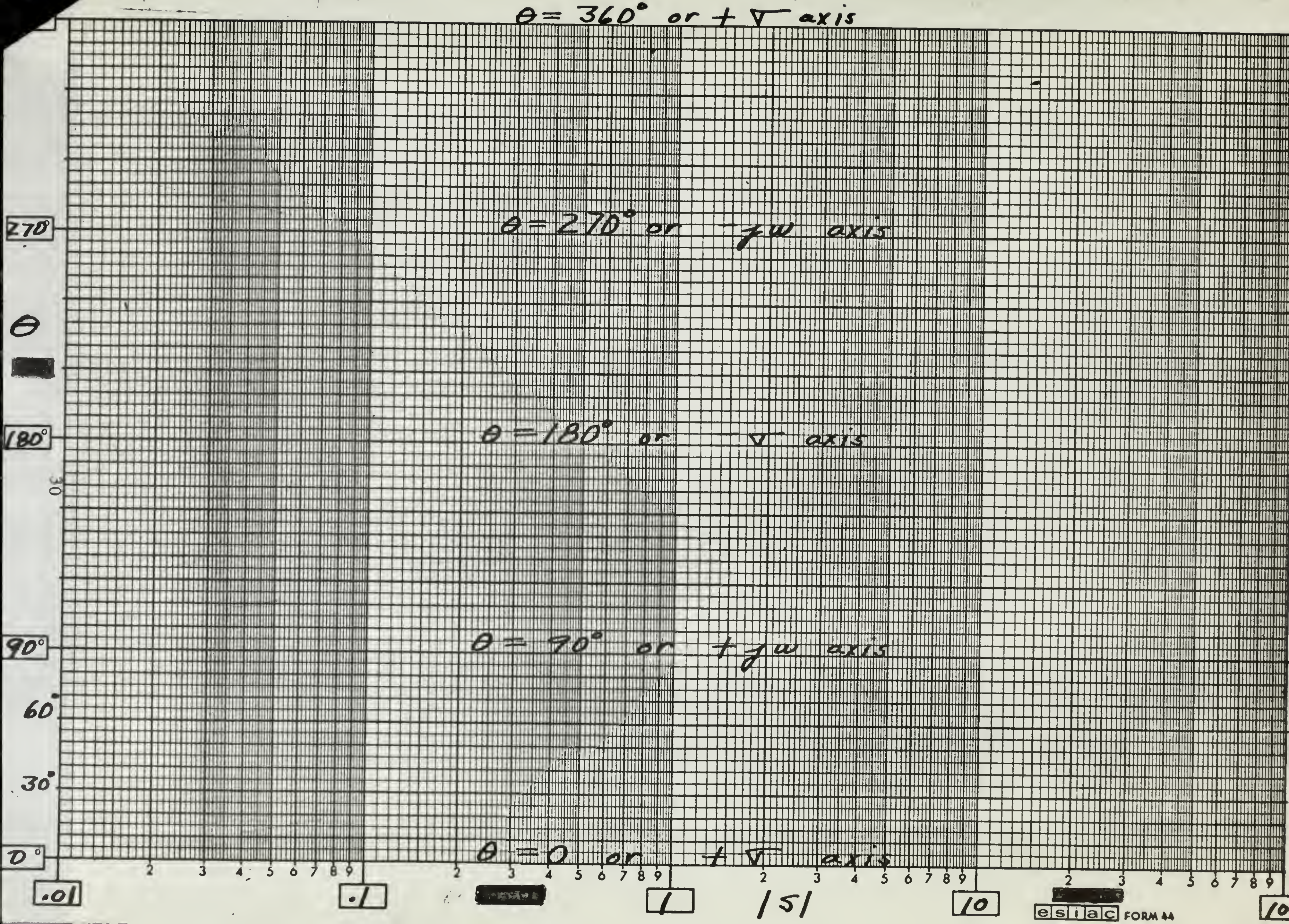

θ = 360° or + σ axis
θ = 270° or − jω axis
θ = 180° or − σ axis
θ = 90° or + jω axis
θ = 0 or + σ axis
θ
270°
180°
90°
60°
30°
0°
.01
.1
1
|s|
10
100
esiac FORM 44

Figure 3

Figure 4

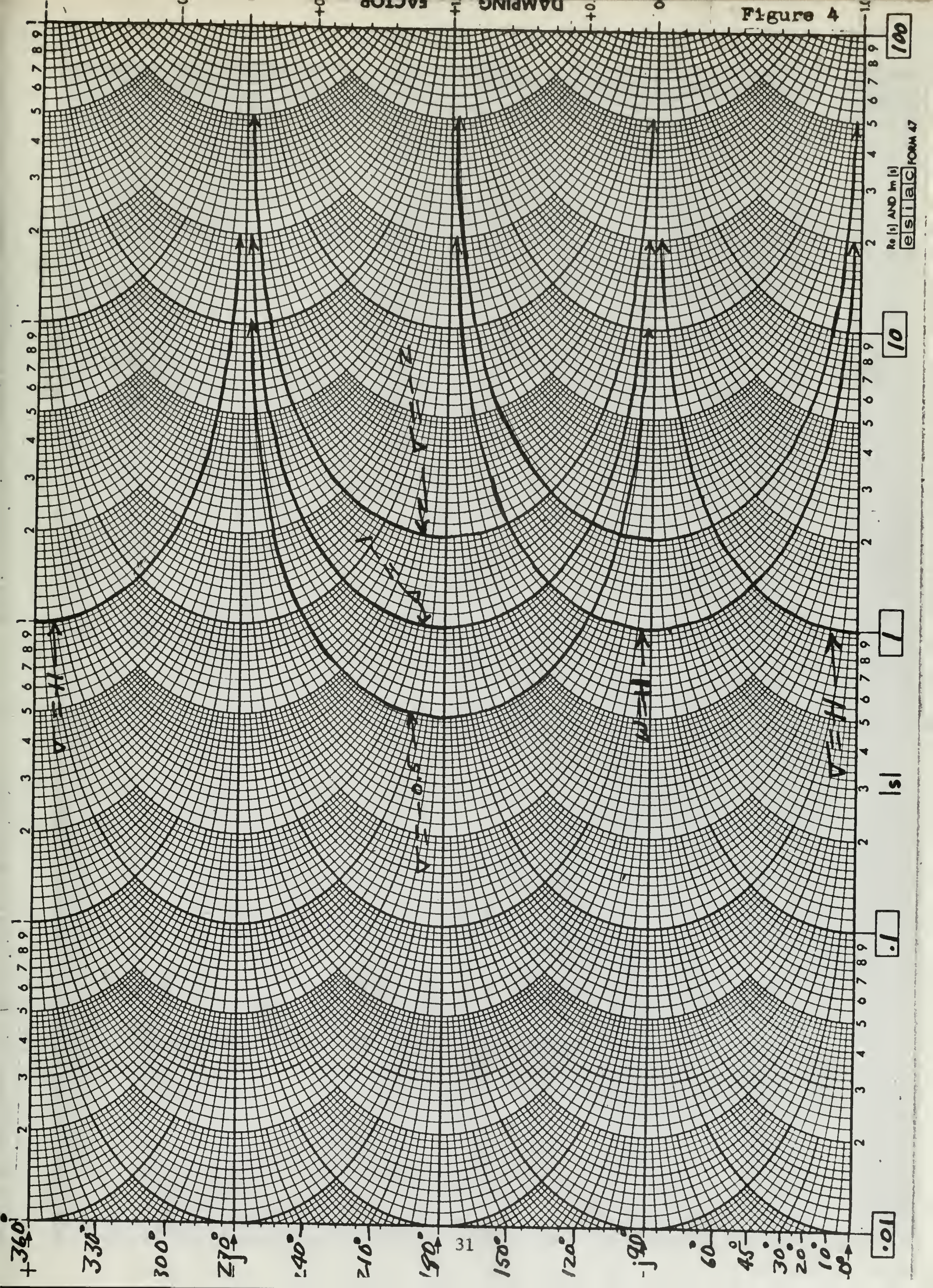
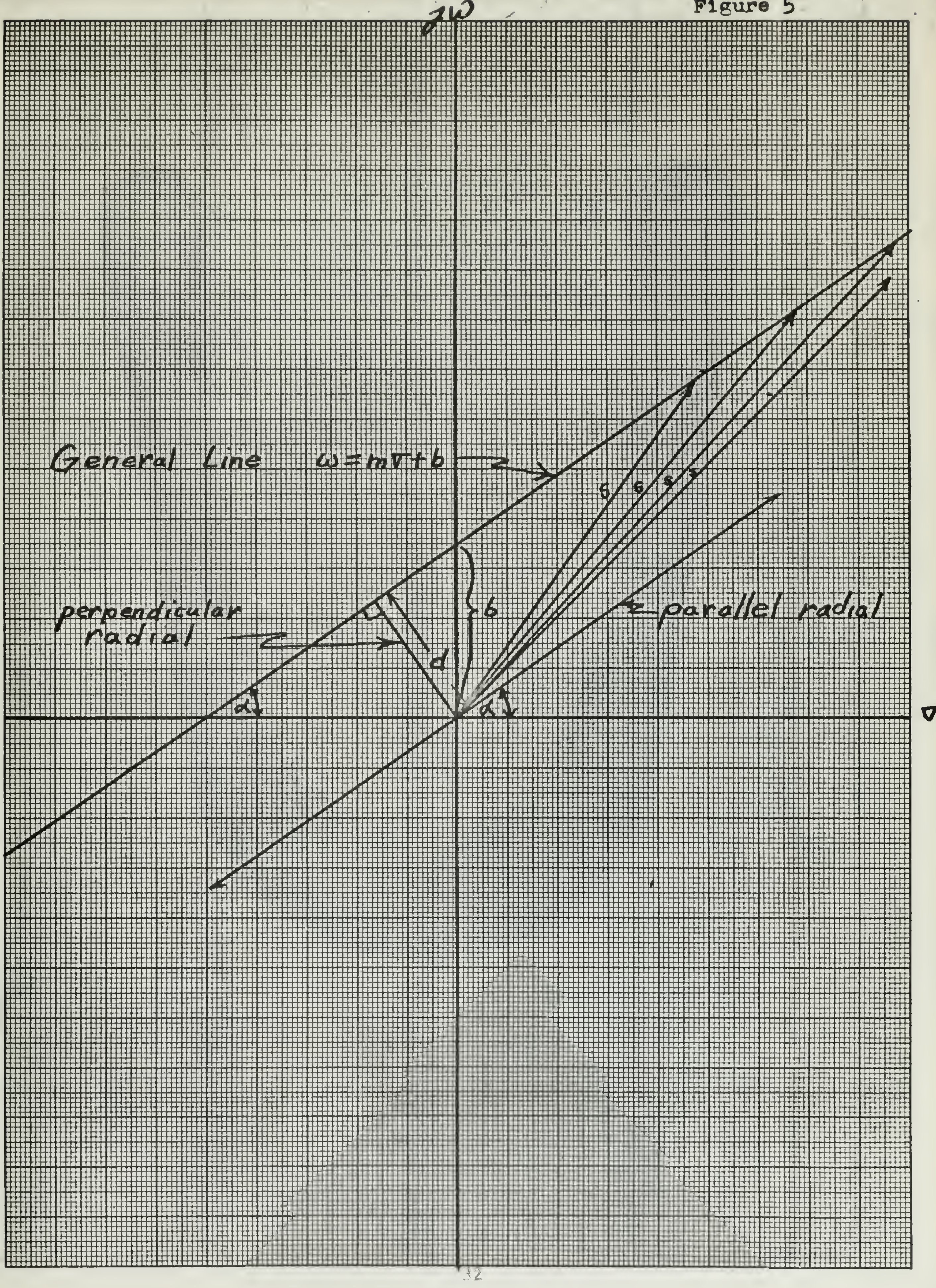
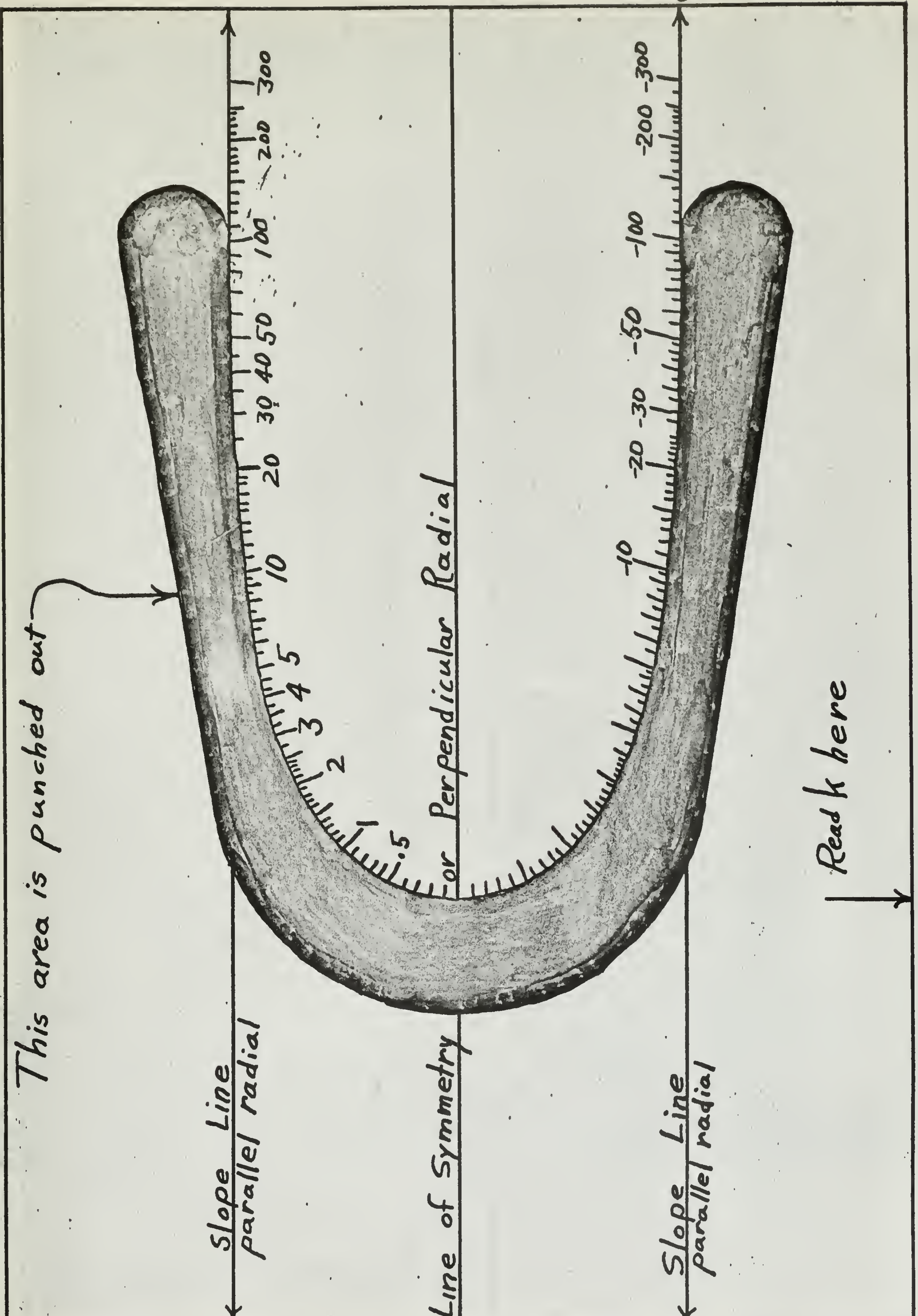

DAMPING FACTOR
|s|
.01
.1
1
10
100
Re[s] AND Im[s]
esiac FORM 47
+360°
330°
300°
270°
240°
210°
180°
150°
120°
-j90°
60°
45°
30°
20°
10°
0°

Figure 5



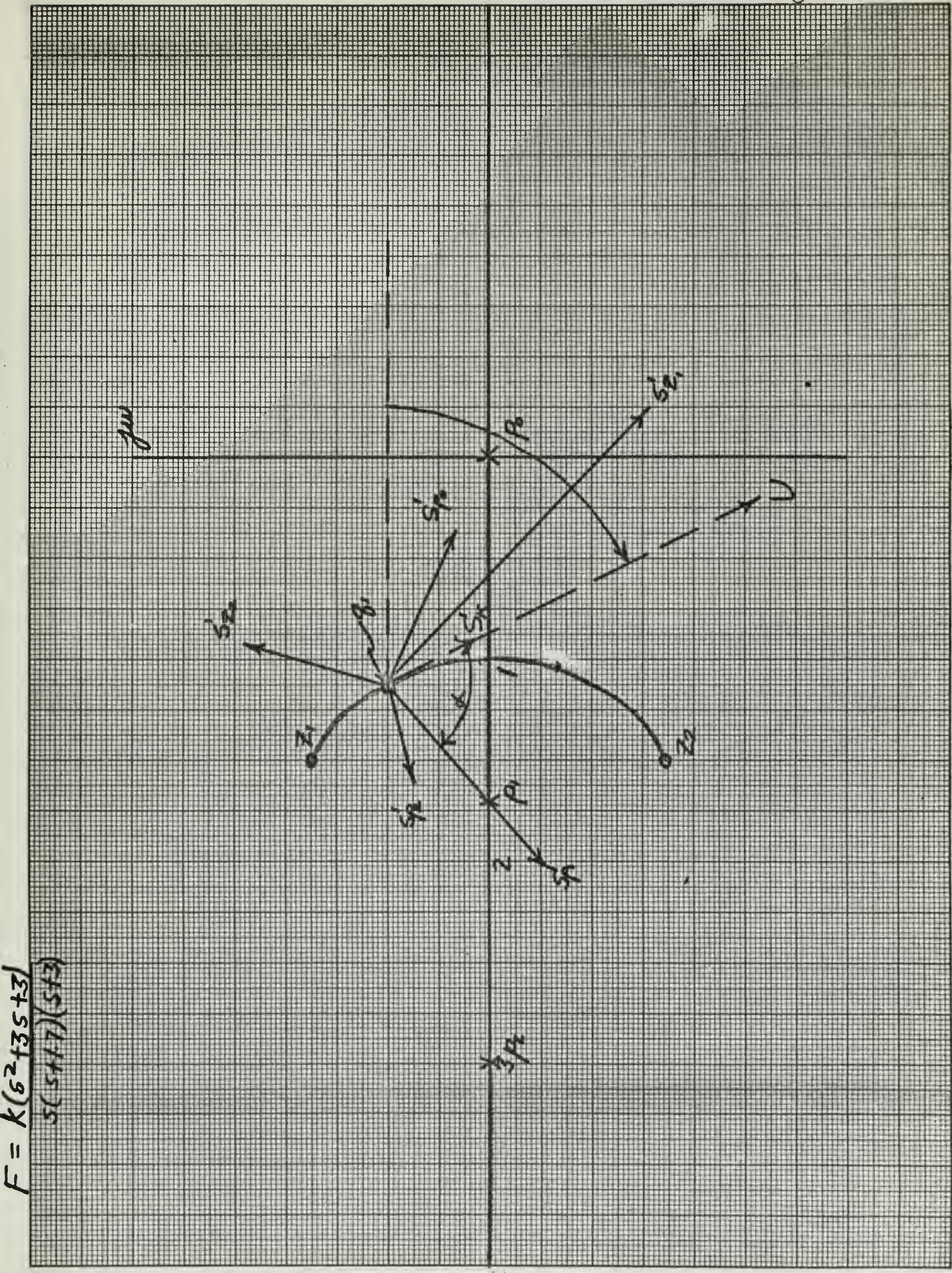
jw
General Line ω=mσ+b
5
6
5
3
b
perpendicular radial
d
parallel radial
α
α
σ

Figure 6



This area is punched out
Slope Line
parallel radial
Line of Symmetry
or Perpendicular Radial
Slope Line
parallel radial
Read k here
.5
1
2
3
4
5
10
20
30
40
50
100
200
300
-10
-20
-30
-50
-100
-200
-300

Figure 7



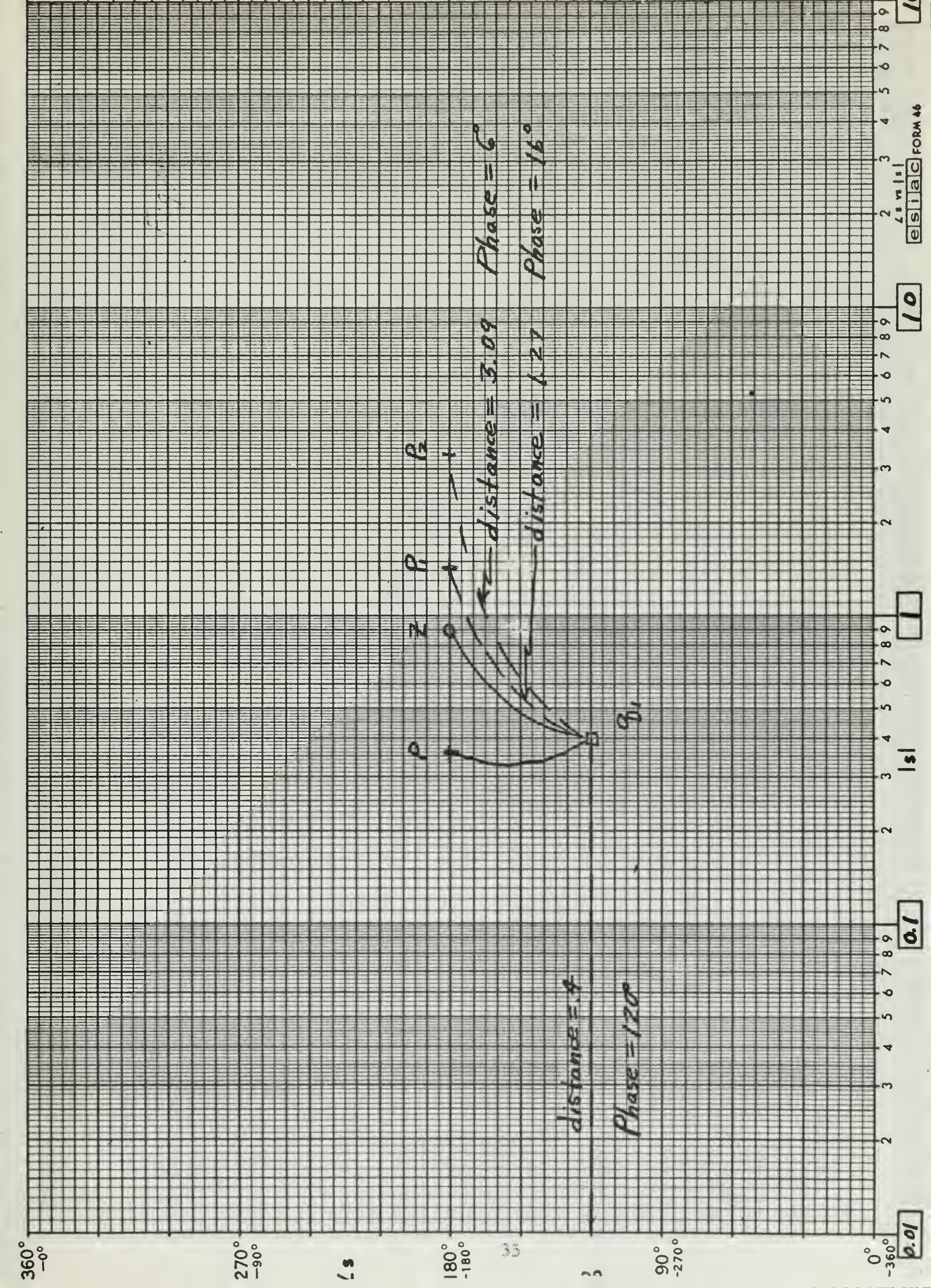
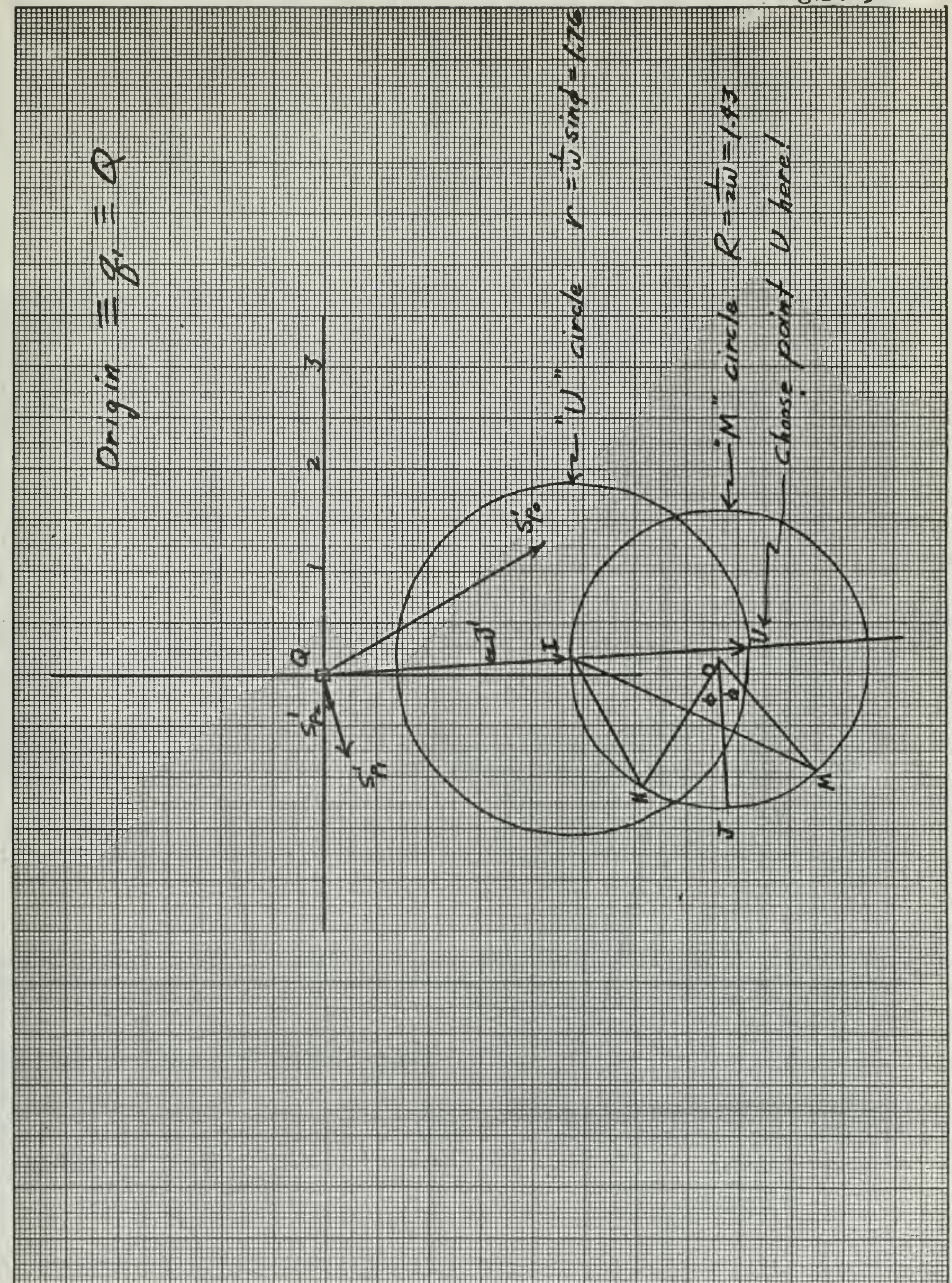

DAMPING FACTOR
|s|
0.01
0.1
1
10
100
360° -0°
270° -90°
180° -180°
90° -270°
0° -360°
esiac FORM 46
P
Z
P1
P2
distance = 3.09   Phase = 6°
distance = 1.27   Phase = 16°
distance = .4
Phase = 120°

Figure 8

Figure 9



Origin ≡ g_1 ≡ Q
"U" circle
$r = \frac{1}{\omega \sin\phi} = 1.76$
"M" circle
$R = \frac{1}{2\omega} = 1.43$
Choose point U here!
1
2
3
Q
I
J
M
M'
U

$$G = \frac{k}{s(s+.355)(s+333)}$$

* "Wheeler" (k vs Re s) plot to determine breakaway point.

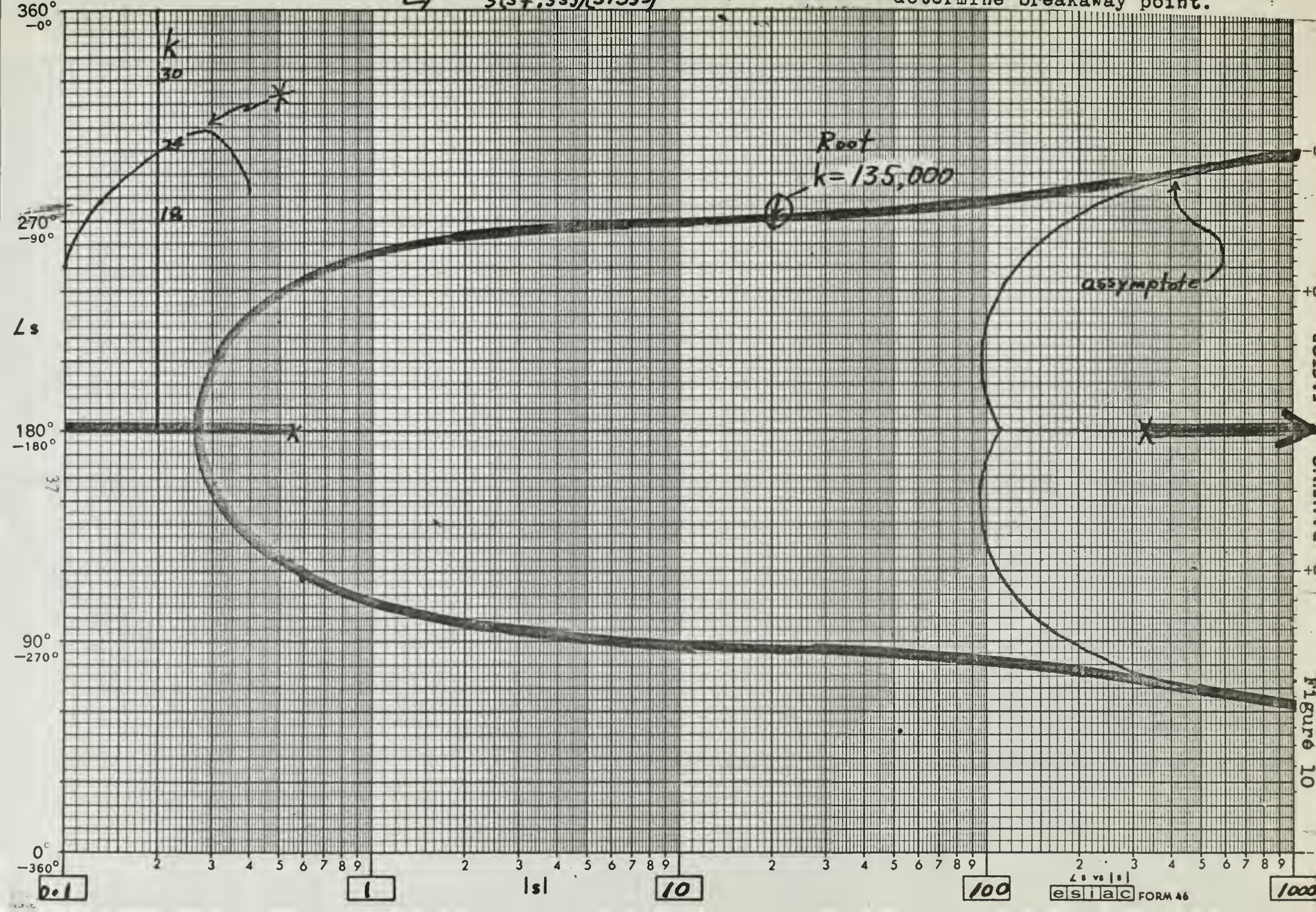


Figure 10

37

esi 7524 S.W. MACADAM AVENUE PORTLAND 19, OREGON

esiac FORM 46

∠s vs |s|

Figure 11

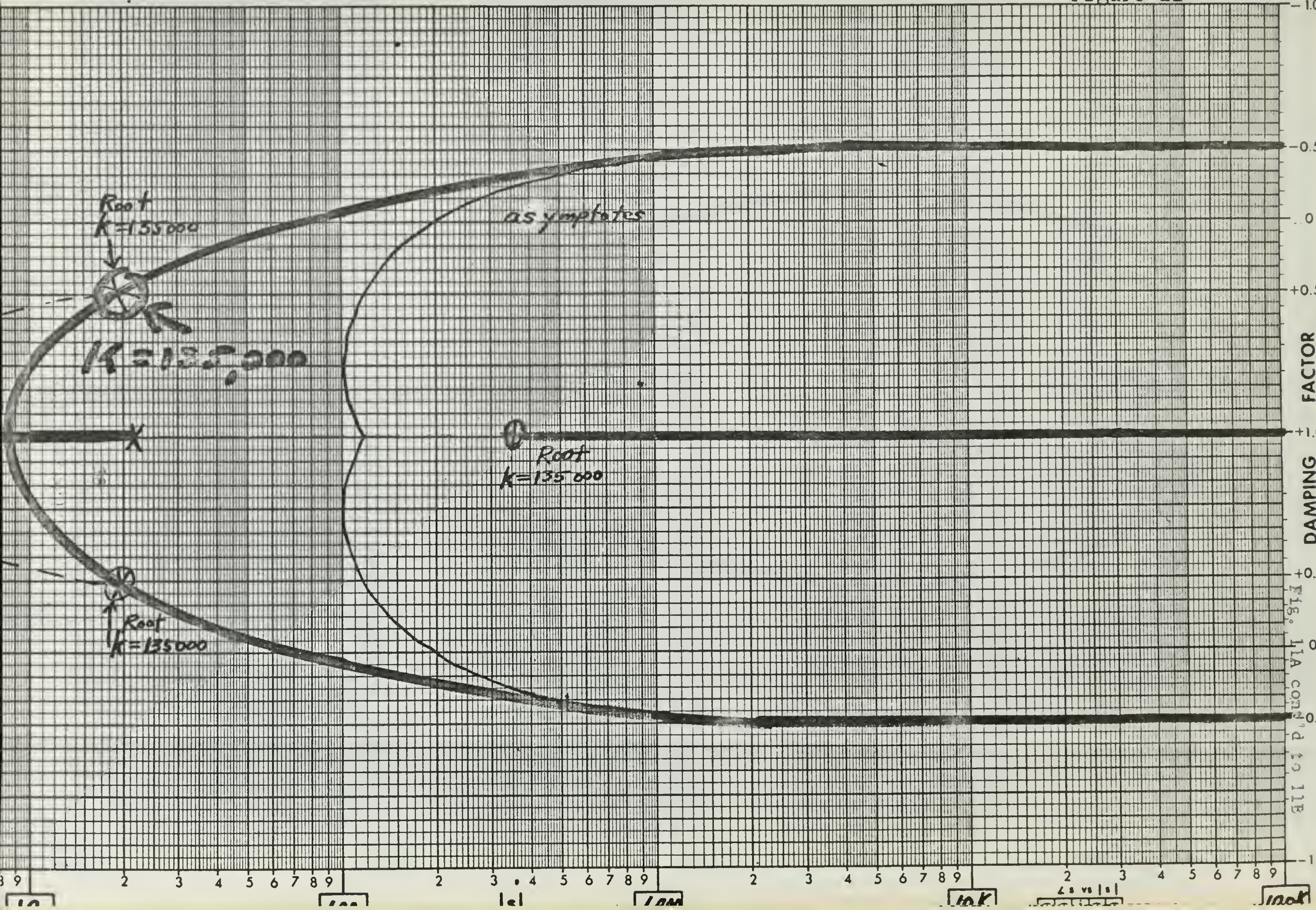
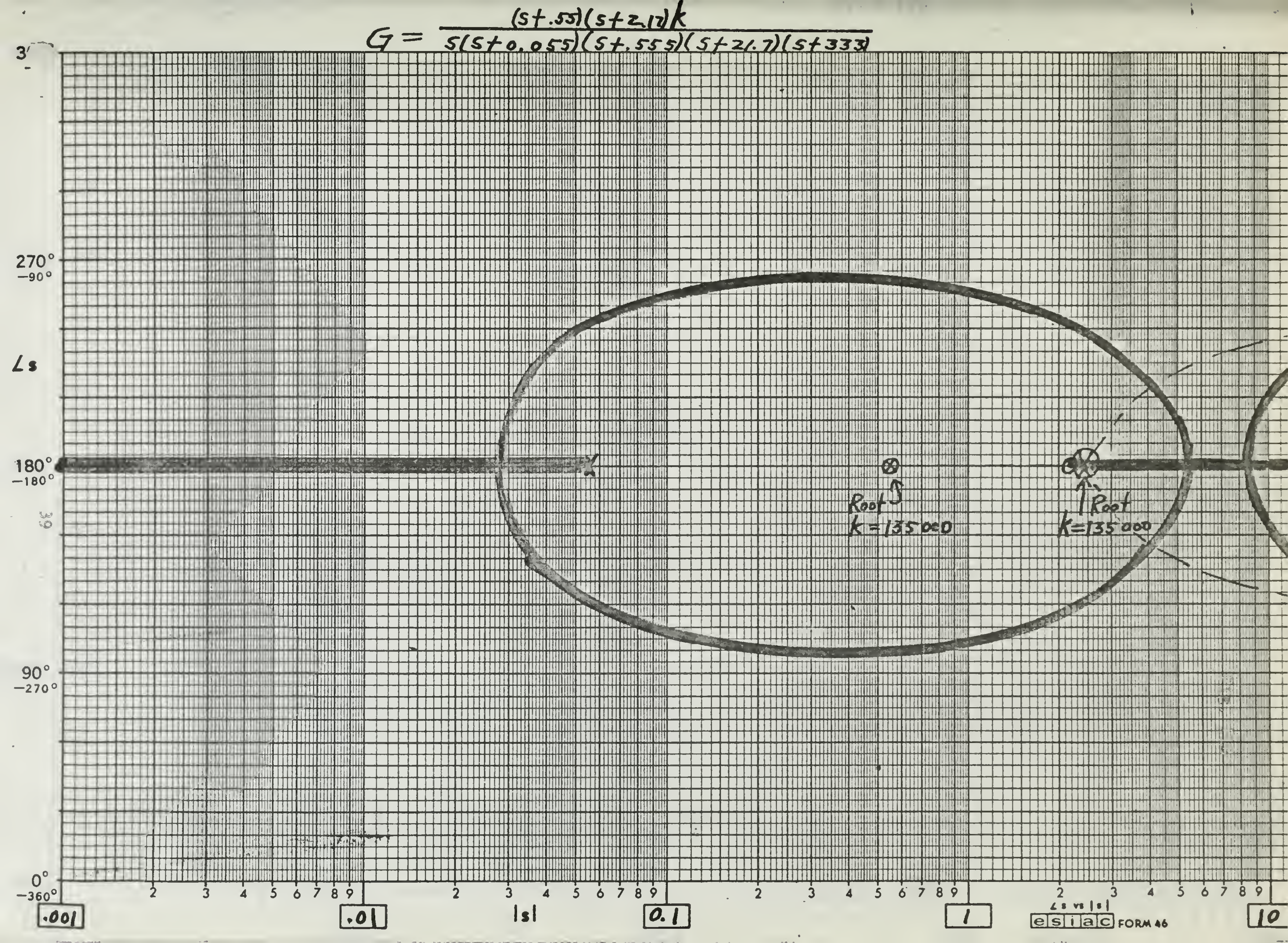
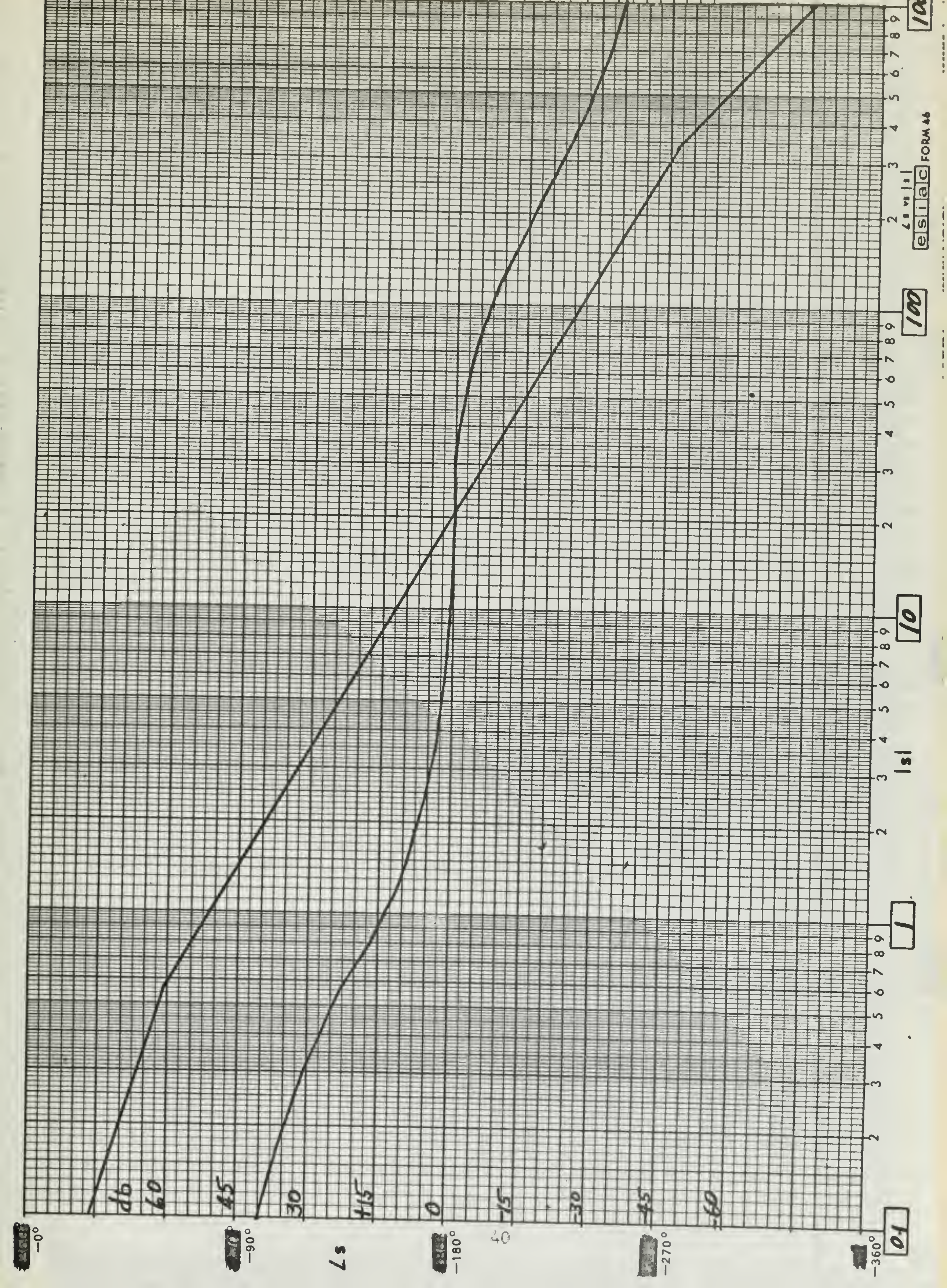


Fig. 11A cont'd to 11B



$$G = \frac{(s+.55)(s+2.12)k}{s(s+0.055)(s+.555)(s+21.7)(s+333)}$$
270° −90°
∠s
180° −180°
90° −270°
0° −360°
Root
k = 135 000
Root
k = 135 000
|s|
.001
.01
0.1
1
10
∠s vs |s|
esiac FORM 46

Figure 12

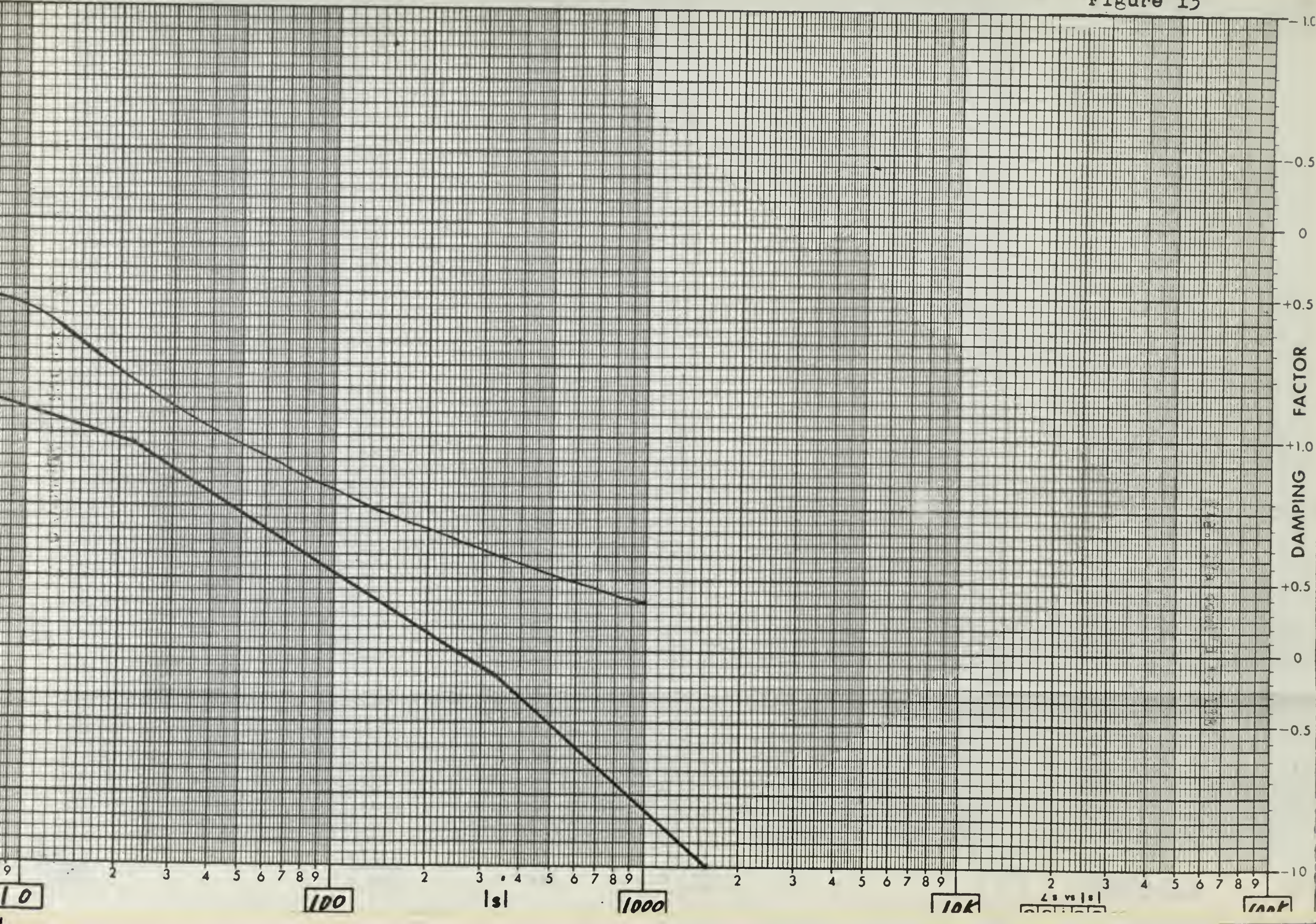


db
60
45
30
+15
0
-15
-30
-45
-60
|s|
.01
1
10
100
1000
-0°
-90°
-180°
-270°
-360°
esiac FORM 46

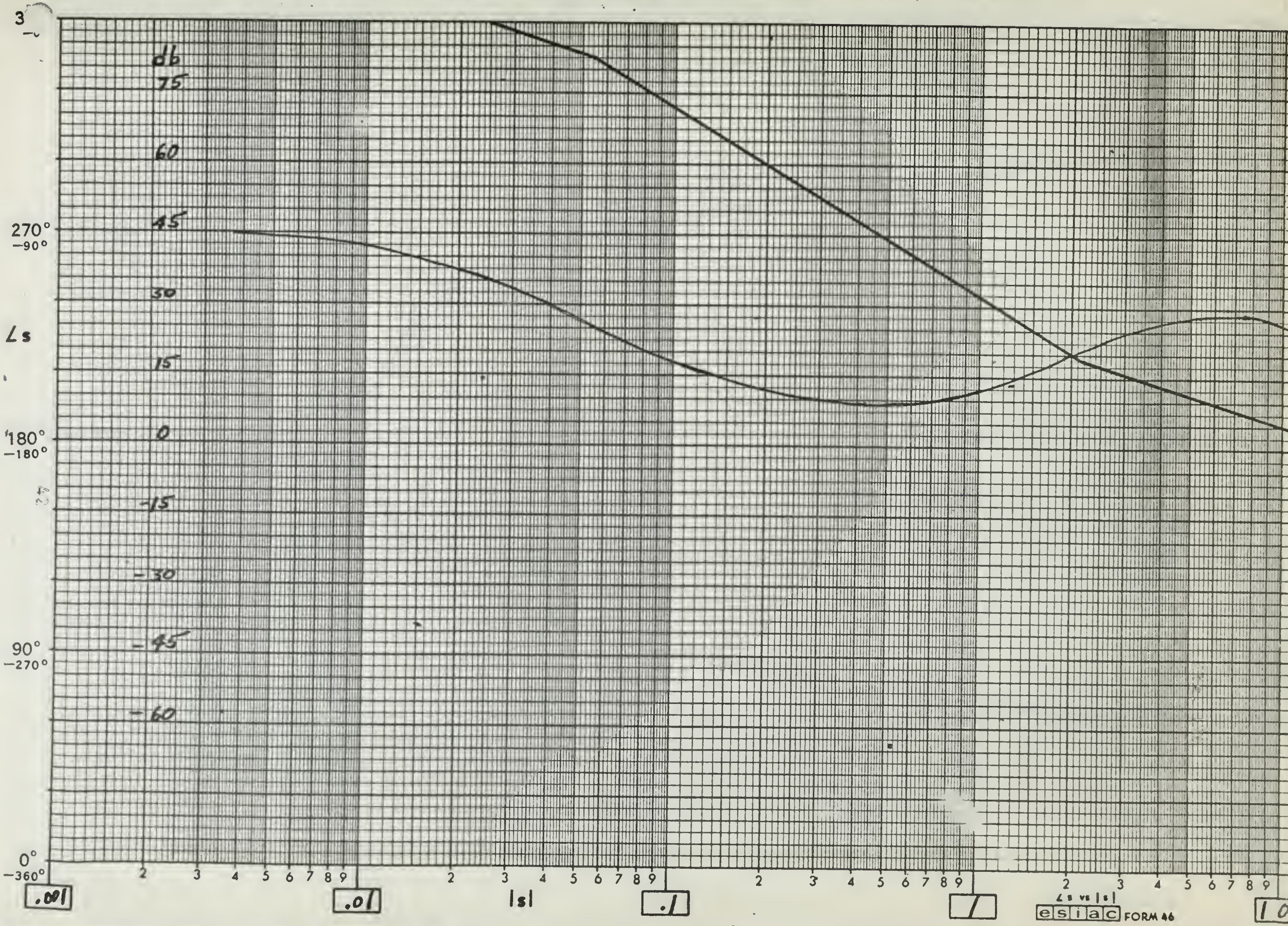
esiac FORM 46

∠s vs |s|

Figure 13



$\angle s$ vs $|s|$



db
75
60
45
30
15
0
-15
-30
-45
-60
270°
-90°
180°
-180°
90°
-270°
0°
-360°
$\angle s$
|s|
.001
.01
.1
1
10
$\angle s$ vs |s|
esiac FORM 46

$$\frac{200}{s(s+49.9)(s+2.1)} = -1$$

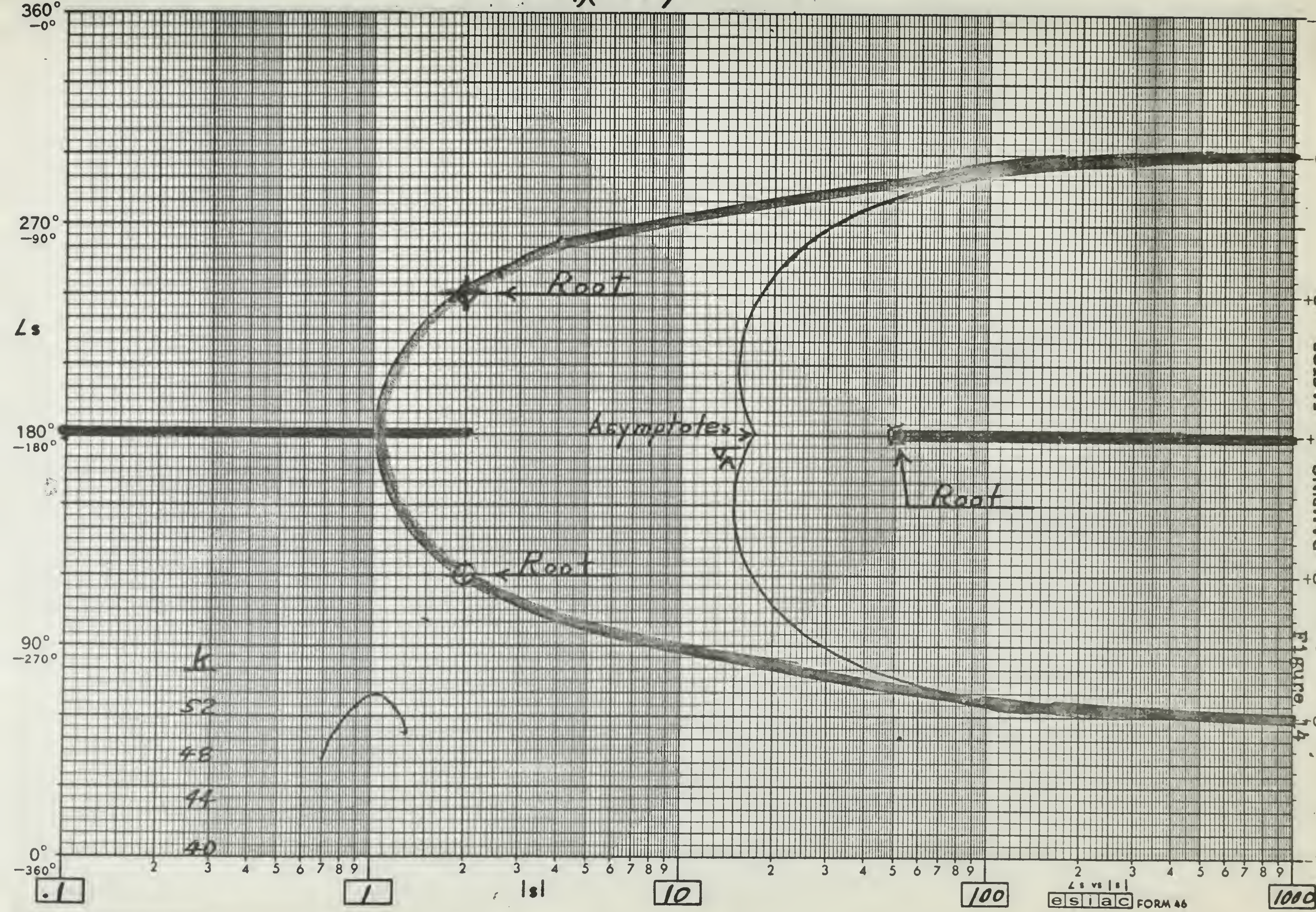


Figure 14

$$\frac{-210}{s(s+9+\angle 2.5)(s+9-\angle 2.5)} = -1$$

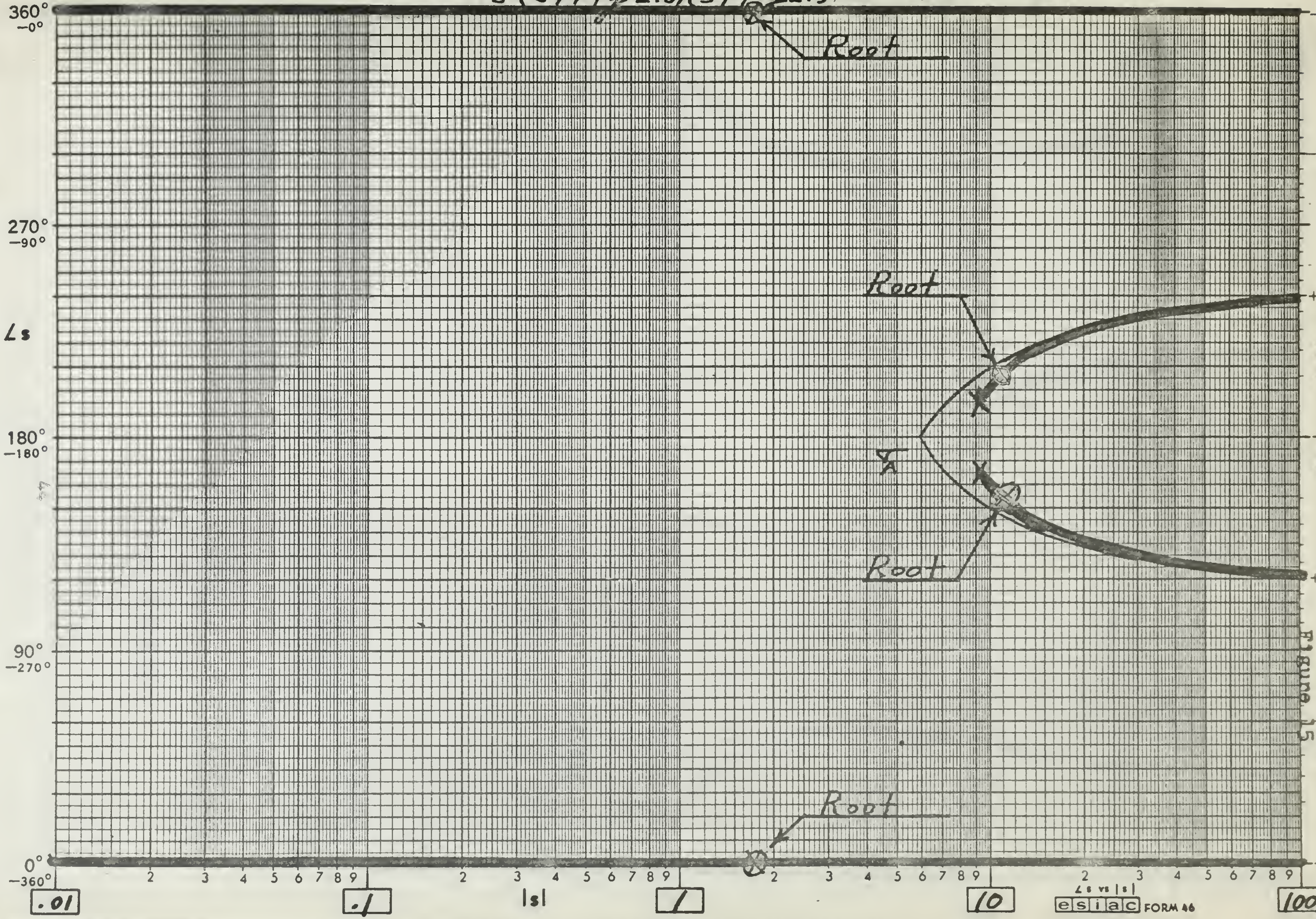
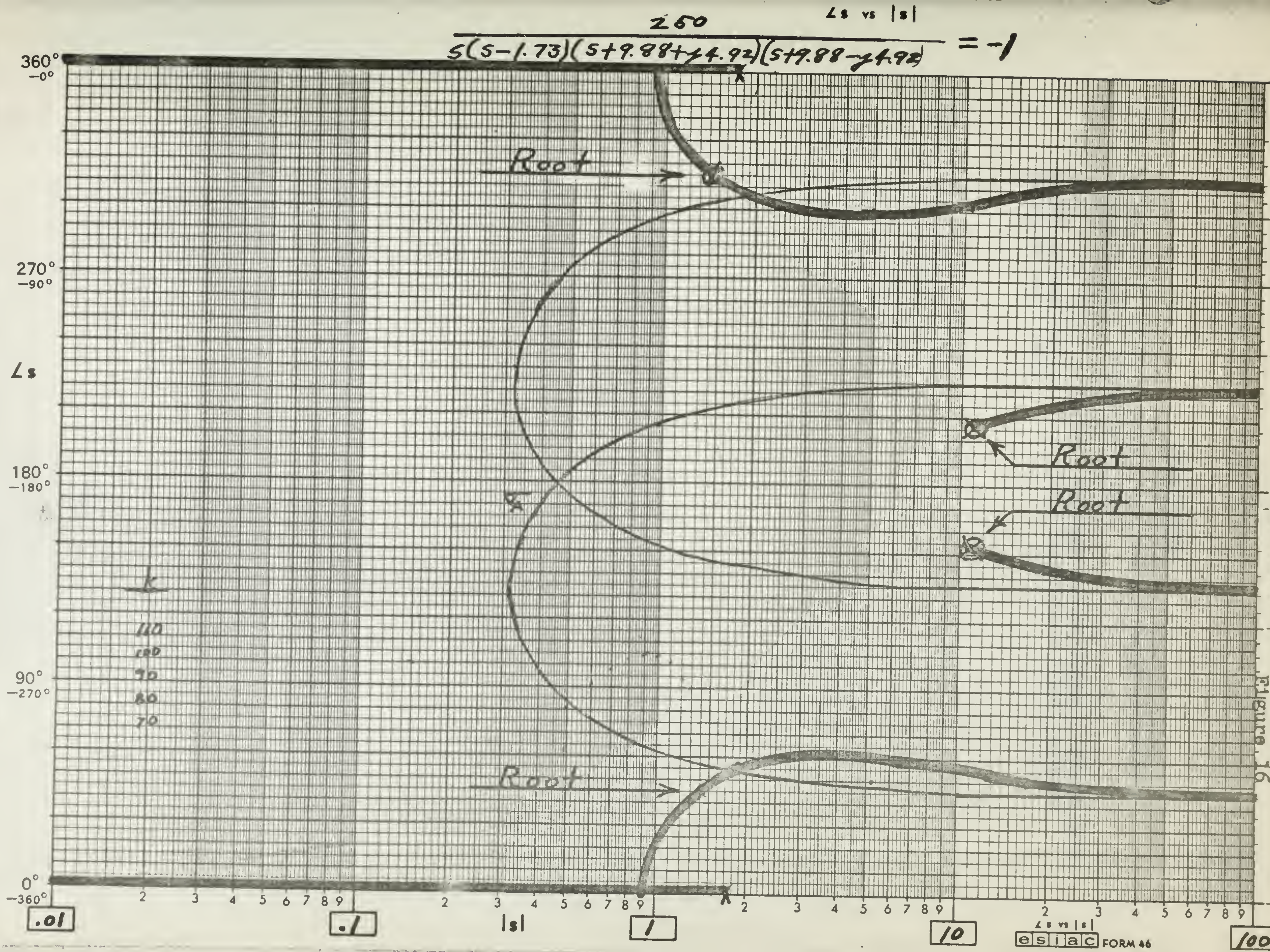


Figure 15



$$\frac{250}{s(s-1.73)(s+9.88+j4.92)(s+9.88-j4.92)} = -1$$
∠s vs |s|
360° −0°
270° −90°
∠s
180° −180°
90° −270°
0° −360°
Root
Root
Root
Root
k
110
100
90
80
70
.01
.1
|s|
1
10
100
∠s vs |s|
esiac FORM 46

Figure 16

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2. Rung, B. T. and G. J. Thaler. Feedback Control Systems: Design with Regard to Sensitivity. U. S. Naval Postgraduate School Research Report No. 41. 1964.